

Prerequisites for the Design Theory of Flying Wings and Tailless Aircraft

A preparatory learning module in mathematics, aerodynamics and flight mechanics for undergraduate aerospace engineering

This module builds, from first principles, every idea that the companion module *Design Theory and Aerodynamics of Flying Wings and Tailless Aircraft* takes for granted. That module assumes a working knowledge of multivariable calculus and of low-speed aerodynamics: lift, drag and pitching moment, airfoil polars, and the basics of finite-wing theory. Here each of those foundations is developed in full. The mathematics comes first, because the language of vectors, partial derivatives, line integrals and Fourier series is used throughout. The physics of the fluid follows: its properties, the standard atmosphere, the conservation laws and Bernoulli's equation. Dimensional analysis then explains why aerodynamic data are presented as coefficients and why the Reynolds number matters so much for small aircraft. The boundary layer explains drag and stall. Airfoil geometry, the integration of surface pressure into forces and moments, potential flow, the Kutta–Joukowski theorem and thin-airfoil theory explain where lift comes from and why the aerodynamic centre sits near the quarter chord. Airfoil polars, drag breakdowns and performance relations follow. The finite wing is treated through vortex theory and Prandtl's lifting-line theory, which is the foundation of every statement about elliptical and bell-shaped lift distributions. Sweep, flight-mechanics conventions, the concept of stability and a minimal treatment of wing bending complete the preparation.

Each section ends with the specific results that the flying-wing module relies upon. Worked examples use a single small electric flying wing throughout, so that the numbers can be followed from section to section. Review questions are collected at the end, together with a map from each prerequisite section to the place where it is used.

How to use this module

Read the sections in order on a first pass, since later sections lean heavily on earlier ones. Work every derivation with pencil and paper rather than reading it, because the algebra in thin-airfoil and lifting-line theory only becomes familiar by repetition. Readers who are already confident with vector calculus may skim Section 1 but should still read Section 1.7 on Fourier series, which is the key to Sections 10 and 14. Readers who already know elementary fluid mechanics may begin at Section 5.

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Notation and Conventions

Units are SI throughout. Angles in formulas are in radians unless degrees are stated. Pitching moments are positive nose-up. Chordwise positions are measured aft from the leading edge and are written either as a length x or as a fraction of the chord, $h = x/c$. Lower-case coefficients (c_p , c_d , c_m) refer to a two-dimensional airfoil section of infinite span. Upper-case coefficients (C_L , C_D , C_m) refer to a finite wing or a complete aircraft. A prime denotes a quantity per unit span, as in L' for lift per unit span. The subscript ∞ denotes the undisturbed free stream far from the body.

Symbol	Meaning	Unit
a	Speed of sound; or finite-wing lift-curve slope (context makes clear)	m/s; 1/rad
a_0	Section (two-dimensional) lift-curve slope, $dc/d\alpha$	1/rad
AR	Aspect ratio, b^2/S	–
b	Wing span	m
$c, \bar{c}$	Local chord; mean aerodynamic chord (MAC)	m
c_r, c_t	Root chord, tip chord	m
C_p	Pressure coefficient, $(p - p_\infty)/q_\infty$	–
C_f, c_f	Average and local skin-friction coefficients	–
C_{D0}, K	Zero-lift drag coefficient; induced-drag factor $1/(\pi AR e)$	–
D, L, N, A	Drag, lift, normal force, axial force	N
e	Span efficiency (Oswald) factor	–
g	Gravitational acceleration, 9.80665	m/s ²
h	Chordwise position as a fraction of chord (subscripts ac, cp, cg, np)	–
M	Pitching moment; or Mach number (context makes clear)	N·m; –
p, p_0	Static pressure; total (stagnation) pressure	Pa
q	Dynamic pressure, $\frac{1}{2}\rho V^2$; or pitch rate	Pa; rad/s
R	Specific gas constant for air, 287.05; or resultant force	J/(kg·K); N
Re	Reynolds number, $\rho V l / \mu$	–
S	Wing reference (planform) area	m ²
T	Temperature; or thrust	K; N
u, v, w	Velocity components; w is also the downwash	m/s
$\mathbf{V}, V$	Velocity vector and its magnitude	m/s
W	Weight	N
x, y, z	Coordinates: x chordwise (aft), y spanwise, z vertical	m
α	Angle of attack	rad
$\alpha_{L=0}$	Zero-lift angle of attack	rad
α_i	Induced angle of attack	rad
β	Sideslip angle	rad
γ	Ratio of specific heats (1.4); vortex-sheet strength; or flight-path angle	–; m/s; rad

Symbol	Meaning	Unit
Γ	Circulation	m^2/s
Γ_D	Dihedral angle	rad
δ	Boundary-layer thickness; or control-surface deflection	m; rad
ε	Twist angle (negative = washout); or downwash angle at a tail	rad
η	Non-dimensional semispan station $y/(b/2)$; or efficiency	–
θ	Glauert angle variable; pitch attitude; or momentum thickness	rad; rad; m
λ	Taper ratio c_t/c_r	–
Λ	Sweep angle (subscript gives the reference line)	rad
μ, ν	Dynamic viscosity; kinematic viscosity μ/ρ	$\text{Pa}\cdot\text{s}$; m^2/s
ρ, σ	Density; density ratio ρ/ρ_0	kg/m^3 ; –
τ	Shear stress	Pa
ϕ, ψ	Velocity potential, stream function; also bank and heading angles	m^2/s ; rad
ω	Vorticity, $\nabla \times \mathbf{V}$	1/s

Table 1. Principal symbols. Further symbols are defined where they first appear.

1. Mathematical Foundations

Aerodynamics is written in the language of vector calculus. Velocity is a vector that varies from point to point, pressure is a scalar that varies from point to point, and the forces on a wing are integrals of these fields over its surface. This section collects the mathematics used in the rest of the module and explains the physical meaning of each operation, because the meaning matters more than the manipulation.

1.1 Scalars, vectors and coordinate systems

A **scalar** has magnitude only: temperature, pressure, density and speed are scalars. A **vector** has magnitude and direction: velocity, force and moment are vectors. In a right-handed Cartesian system with unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ along the x , y and z axes, a vector is written by its components,

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}, \quad |\mathbf{V}| = \sqrt{u^2 + v^2 + w^2}$$

A right-handed system is one in which curling the fingers of the right hand from x toward y makes the thumb point along z . Aircraft use several right-handed systems at once, as described in Section 16, and much confusion in stability work is caused simply by mixing them.

The **dot product** (scalar product) of two vectors is a scalar,

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

where θ is the angle between them. Geometrically it is the length of one vector multiplied by the projection of the other onto it. It is used whenever a component in a given direction is wanted: the component of the free stream normal to a swept leading edge, the work done by a force, or the flow rate of fluid through a surface. Two vectors are perpendicular when their dot product is zero.

The **cross product** (vector product) is a vector perpendicular to both of its arguments,

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

with direction given by the right-hand rule. The most important use in this module is the **moment** of a force $\mathbf{F}$ applied at position $\mathbf{r}$ relative to a reference point O , $\mathbf{M} = \mathbf{r} \times \mathbf{F}$. Only the component of the force perpendicular to the lever arm produces a moment, which is why the cross product contains $\sin \theta$. Every pitching-moment calculation in the flying-wing module, including the transfer of moments between the aerodynamic centre and the centre of gravity, is an application of this definition.

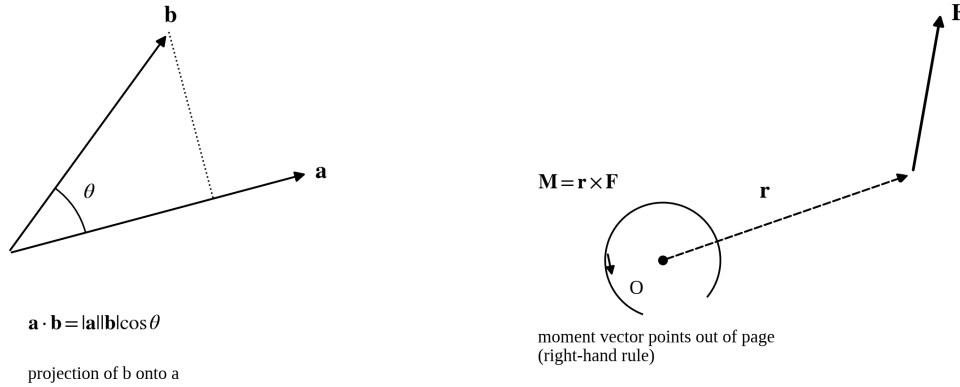


Figure 1. Left: the dot product as a projection. Right: the moment of a force about a point O is the cross product of the position vector and the force, and points along the axis of rotation by the right-hand rule.

1.2 Functions of several variables and partial derivatives

A field quantity such as pressure depends on position and, in general, time: $p = p(x, y, z, t)$. The **partial derivative** $\partial p / \partial x$ is the rate of change of p with x while all the other variables are held fixed.

Aerodynamic coefficients are also functions of several variables. The lift coefficient of a wing depends on the angle of attack, the Mach number, the Reynolds number and the control deflections, and each partial derivative, such as $\partial C_L / \partial \alpha$, measures sensitivity to one of them.

The **total differential** combines the separate effects of small changes in every variable:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial t} dt$$

Dividing by dt along a path $x(t), y(t), z(t)$ gives the **chain rule**, which in Section 3 becomes the material derivative. The **product rule**, $d(uv) = u dv + v du$, is used in Section 3.2 of the flying-wing module to differentiate the moment $C_L(h_{cp} - h_{cg})$, where both the lift and the centre-of-pressure position change with angle of attack. The **quotient rule**, $d(u/v) = (v du - u dv)/v^2$, gives the rate of travel of the centre of pressure from $h_{cp} = h_{ac} - c_{m,ac}/c_l$.

1.3 Linearisation and Taylor series

A smooth function can be approximated near a chosen point x_0 by its **Taylor series**,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f'''(x_0)(x - x_0)^3 + \dots$$

For a function of several variables the first-order terms are the partial derivatives:

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$$

Keeping only the first-order terms is called **linearisation**, and it underlies nearly all of classical aerodynamics and flight mechanics. Over a useful range of angles of attack the lift coefficient is very nearly a straight line, $c_l = a_0(\alpha - \alpha_{L=0})$, and the pitching moment is very nearly a straight line in α as well. The slopes of these straight lines are the **stability derivatives**, such as $C_{m,\alpha} = \partial C_m / \partial \alpha$, which are nothing more than the first-order Taylor coefficients of the aerodynamic forces and moments about a trimmed flight condition.

The small-angle approximations are first-order Taylor series of the trigonometric functions about zero: $\sin \alpha \approx \alpha$, $\cos \alpha \approx 1$ and $\tan \alpha \approx \alpha$, with α in radians. At 10° (0.1745 rad) the error in $\sin \alpha \approx \alpha$ is only 0.5 percent, and the error in $\cos \alpha \approx 1$ is 1.5 percent. These approximations are what allow lift to be treated as equal to the normal force and allow moments to be transferred along the chord with the simple formulas used in Section 8.

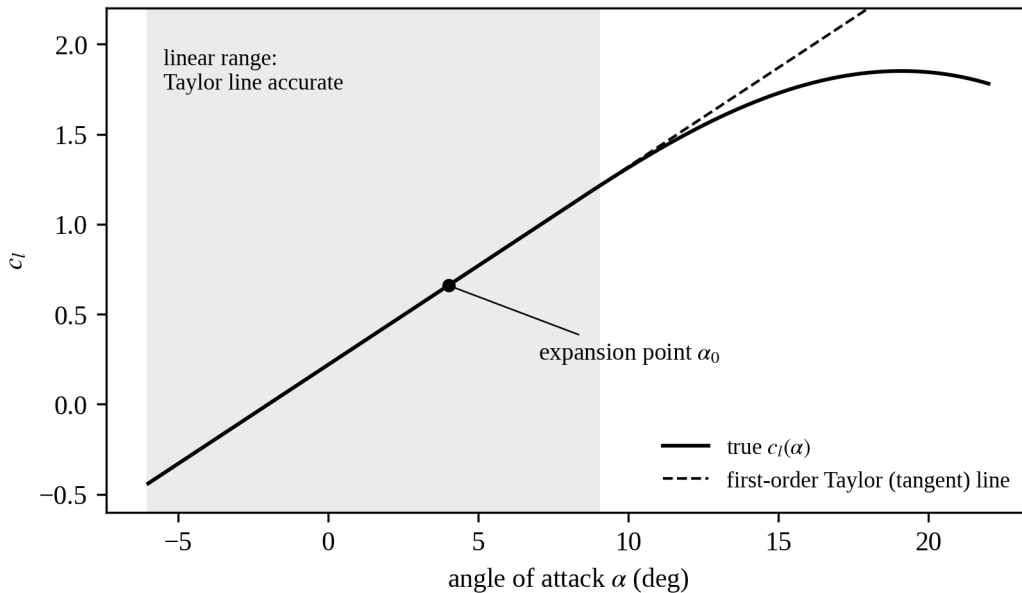


Figure 2. Linearisation. Near the expansion point the first-order Taylor line follows the true lift curve closely. It fails approaching stall, which is why linear theories cannot predict the maximum lift coefficient.

Angles must be in radians whenever they appear in a formula derived from calculus. Lift-curve slopes are nevertheless often quoted per degree. Since one radian is 57.30° , the thin-airfoil slope of 2π per radian is 0.1097 per degree.

1.4 Scalar and vector fields: gradient, divergence and curl

The vector differential operator nabla (del) is

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

It produces three operations, each with a definite physical meaning.

The **gradient** of a scalar field, ∇p , is a vector that points in the direction in which p increases fastest, with magnitude equal to that rate of increase. It is perpendicular to the surfaces of constant p . The net pressure force per unit volume on a fluid element is $-\nabla p$: fluid is pushed from high pressure toward

low pressure. The rate of change of p in the direction of a unit vector $\mathbf{n}$ is $\nabla p \cdot \mathbf{n}$, the directional derivative.

$$\nabla p = \frac{\partial p}{\partial x} \mathbf{i} + \frac{\partial p}{\partial y} \mathbf{j} + \frac{\partial p}{\partial z} \mathbf{k}$$

The **divergence** of a vector field, $\nabla \cdot \mathbf{V}$, is a scalar that measures the net rate of outflow per unit volume from an infinitesimal region. A positive divergence means the region acts as a source. For a fluid of constant density, mass conservation requires $\nabla \cdot \mathbf{V} = 0$ everywhere (Section 3.3).

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The **curl** of a vector field, $\nabla \times \mathbf{V}$, is a vector that measures local rotation. For a velocity field it is called the **vorticity** ω , and it equals twice the angular velocity of a small fluid element. In two-dimensional flow in the x - z plane only one component survives. Vorticity is the central concept of the theory of lift: an airfoil can be modelled as a sheet of vorticity, and a finite wing as a system of vortex lines.

$$\nabla \times \mathbf{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathbf{k}$$

The **Laplacian** is the divergence of the gradient, $\nabla^2 \phi = \partial^2 \phi / \partial x^2 + \partial^2 \phi / \partial y^2 + \partial^2 \phi / \partial z^2$. Laplace's equation, $\nabla^2 \phi = 0$, governs incompressible irrotational flow (Section 3.6). Two identities are used repeatedly: the curl of any gradient is zero, $\nabla \times (\nabla \phi) = 0$, and the divergence of any curl is zero, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

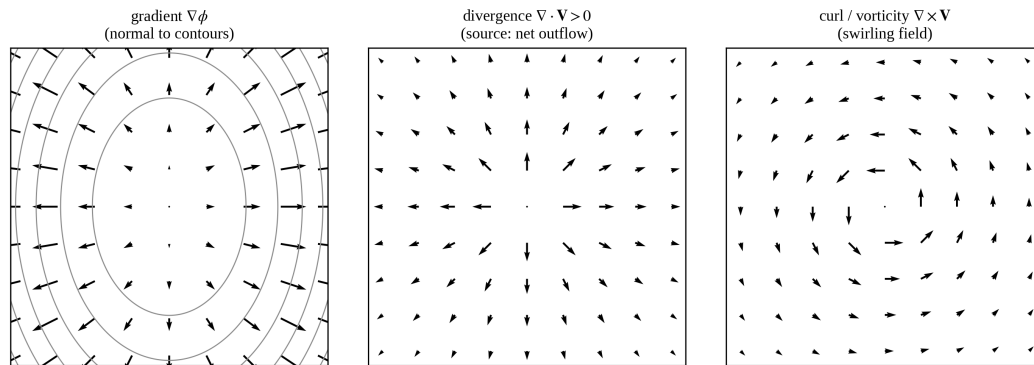


Figure 3. Three operations of the nabla operator. The gradient points across contours toward higher values. A field with positive divergence flows outward from a source. A field with non-zero curl swirls.

1.5 Line, surface and volume integrals

A **line integral** adds up the component of a vector field along a curve C . With ds a small element of the curve, $\int_C \mathbf{V} \cdot d\mathbf{s}$ is the integral of the tangential component of velocity along C . When C is closed, written $\oint_C$, this integral is the **circulation**, the defining quantity of lift. A **surface integral** $\int_S \mathbf{V} \cdot \mathbf{n} dS$ adds up the normal component of a field over a surface with unit normal $\mathbf{n}$, and for a velocity field it gives the volume flow rate through the surface (the flux). A **volume integral** $\int_V \rho dV$ gives, for example, the mass contained in a region. The aerodynamic force on a wing is the surface integral of pressure and shear stress over the wing's surface (Section 8.1).

1.6 The integral theorems

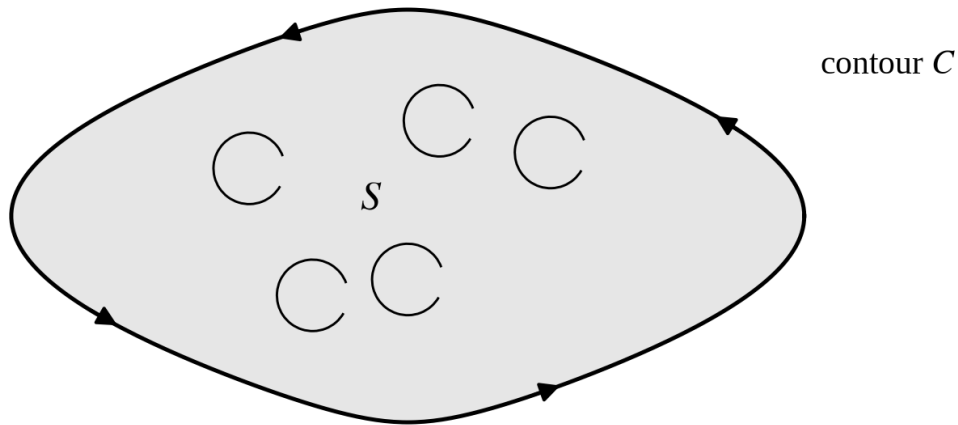
Three theorems connect integrals over a region with integrals over its boundary. Each turns a statement about a point into a statement about a finite region, which is how the differential and integral forms of the conservation laws are related. **Gauss's divergence theorem** states that the total outflow through a closed surface equals the integral of the divergence over the enclosed volume:

$$\iint_S \mathbf{V} \cdot \mathbf{n} dS = \iiint_V \nabla \cdot \mathbf{V} dV$$

Stokes's theorem states that the circulation around a closed curve equals the flux of vorticity through any surface bounded by that curve:

$$\oint_C \mathbf{V} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{V}) \cdot \mathbf{n} dS$$

Green's theorem is the two-dimensional form of Stokes's theorem. Stokes's theorem has a direct aerodynamic consequence. If a flow is irrotational everywhere inside a contour, the circulation around that contour is zero. An airfoil can carry circulation, and therefore lift, only because the contour around it encloses the airfoil itself, where the vorticity is concentrated in the boundary layer, or in the mathematical model, in a bound vortex sheet.



$$\oint_C \mathbf{V} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{V}) \cdot \mathbf{n} dS$$

Figure 4. Stokes's theorem. The circulation around the closed contour C equals the total vorticity passing through the surface S that it bounds.

1.7 Fourier series

A function defined on $0 \leq \theta \leq \pi$ can be represented by a **Fourier sine series**,

$$f(\theta) = \sum_{n=1}^{\infty} A_n \sin n\theta, \quad A_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin n\theta d\theta$$

The coefficient formula follows from the **orthogonality** of the sine functions on this interval:

$$\int_0^{\pi} \sin m\theta \sin n\theta d\theta = 0 \quad (m \neq n), \quad \int_0^{\pi} \sin^2 n\theta d\theta = \frac{\pi}{2}$$

A cosine series is built in the same way, with $\int_0^\pi \cos m\theta \cos n\theta d\theta = \pi/2$ for $m = n \geq 1$. Fourier series appear twice in this module, and in both cases they appear together with a change of variable. In thin-airfoil theory the chordwise position is written $x = (c/2)(1 - \cos \theta)$, and the camber-line slope is expanded in a cosine series in θ . In lifting-line theory the spanwise position is written $y = -(b/2) \cos \theta$, and the circulation is expanded in a sine series in θ . This substitution, due to Glauert, clusters points near the edges where the solution changes rapidly, and it turns awkward integrals into the standard results

$$\int_0^\pi \frac{\cos n\theta}{\cos \theta - \cos \theta_0} d\theta = \frac{\pi \sin n\theta_0}{\sin \theta_0}$$

which is known as the Glauert integral. A symmetric spanwise loading contains only odd harmonics ($n = 1, 3, 5, \dots$), because the even harmonics are antisymmetric about mid-span. The first harmonic alone is the elliptical distribution. The identity $\sin^3 \theta = (3 \sin \theta - \sin 3\theta)/4$ shows that adding a negative third harmonic to the elliptical loading produces the bell-shaped loading discussed in the flying-wing module.

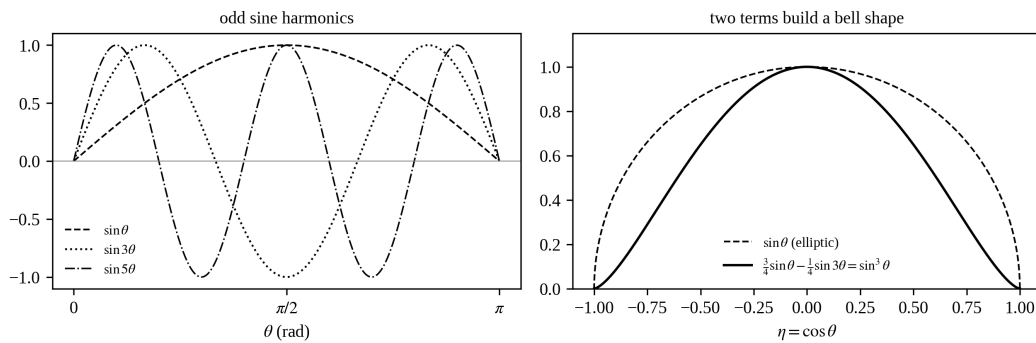


Figure 5. Left: the first odd sine harmonics on $0 \leq \theta \leq \pi$. Right: the elliptical loading is the single harmonic $\sin \theta$; subtracting a third harmonic unloads the tips and produces the bell shape $\sin^3 \theta$.

1.8 What the flying-wing module uses from this section

Partial derivatives and linearisation define the stability derivatives and the lift-curve slope. The cross product defines every moment. The product and quotient rules derive the centre-of-pressure travel. Line integrals and Stokes's theorem define circulation and connect it to vorticity. Fourier series in the Glauert variable are the tool for thin-airfoil theory and for comparing the elliptical and bell-shaped distributions.

2. Properties of the Fluid and the Standard Atmosphere

2.1 The continuum hypothesis

Air consists of molecules, but at sea level a cube of air one micrometre on a side still contains about 2.5×10^7 of them, and the mean free path between collisions is about 68 nanometres. For any aircraft the flow can therefore be treated as a **continuum**: a substance whose properties (density, pressure, velocity, temperature) are defined at every point and vary smoothly. A **fluid element** is a region small enough that its properties are uniform but large enough to contain very many molecules. The continuum hypothesis fails only in very rarefied flow, at the edge of space, which is irrelevant here.

2.2 State variables and the equation of state

The **static pressure** p is the normal force per unit area exerted by the fluid on a surface moving with it, caused by molecular collisions. It acts equally in all directions at a point. The **density** ρ is mass per unit volume. The **temperature** T measures the mean kinetic energy of molecular motion. For air at the conditions met in flight these are related by the equation of state of a perfect gas,

$$p = \rho RT, \quad R = 287.05 \text{ J kg}^{-1} \text{ K}^{-1}$$

so only two of the three state variables can be chosen independently. Pressure and temperature are measured, and density, which appears in every aerodynamic force, is calculated from them.

2.3 Viscosity

A real fluid resists shearing. When adjacent layers slide past each other, momentum is transferred between them by molecular motion, and a tangential stress results. For air, as for water, the shear stress is proportional to the rate of shear. This is Newton's law of viscosity,

$$\tau = \mu \frac{\partial u}{\partial y}$$

where μ is the **dynamic viscosity**. The **kinematic viscosity** $\nu = \mu/\rho$ appears in the Reynolds number. At sea level $\mu = 1.789 \times 10^{-5} \text{ Pa}\cdot\text{s}$ and $\nu = 1.461 \times 10^{-5} \text{ m}^2/\text{s}$. Viscosity of a gas increases with temperature, a dependence described by Sutherland's law,

$$\mu = \mu_{ref} \left(\frac{T}{T_{ref}} \right)^{3/2} \frac{T_{ref} + S}{T + S}, \quad S = 110.4 \text{ K}$$

The most important consequence of viscosity is the **no-slip condition**: the fluid in contact with a solid surface has zero velocity relative to it. Between the wall and the outer flow the velocity rises through a thin region of intense shear, the boundary layer (Section 6). The shear stress at the wall, τ_w , is the origin of skin-friction drag.

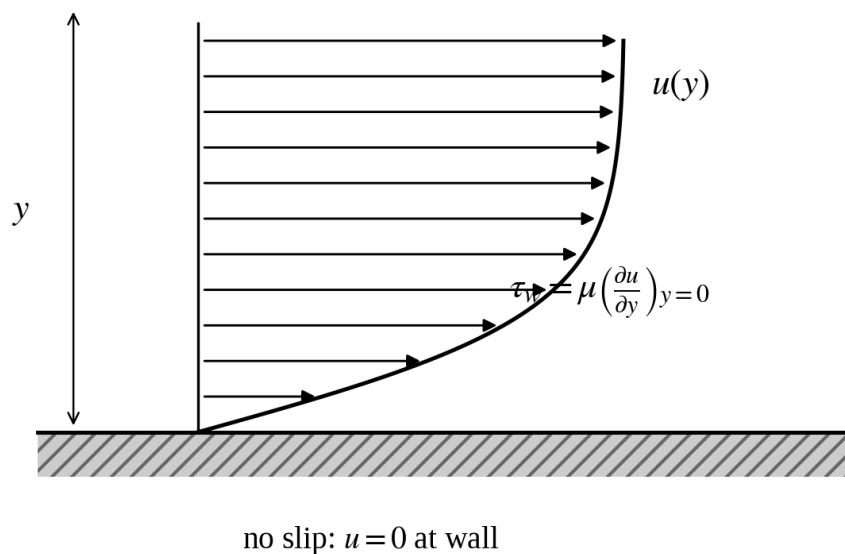


Figure 6. Velocity profile near a wall. The velocity is zero at the surface (no slip) and the wall shear stress is proportional to the velocity gradient there.

2.4 The International Standard Atmosphere

Because aircraft performance depends on air density, a reference atmosphere is needed against which aircraft can be compared. The International Standard Atmosphere (ISA) defines sea-level conditions of $T_0 = 288.15 \text{ K}$ (15°C), $p_0 = 101\,325 \text{ Pa}$ and $\rho_0 = 1.225 \text{ kg/m}^3$. In the **troposphere**, from sea level to 11 km, temperature falls linearly with a lapse rate $L = 0.0065 \text{ K/m}$:

$$T = T_0 - Lh$$

Combining the hydrostatic equation, $dp = -\rho g dh$, with the equation of state and integrating gives

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{g/(LR)} = \left(\frac{T}{T_0}\right)^{5.256}, \quad \frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{4.256}$$

Above 11 km, in the lower **stratosphere**, the temperature is constant at 216.65 K and the pressure falls exponentially, $p = p_{11} \exp[-g(h - 11\,000)/(RT)]$. The ratio $\sigma = \rho/\rho_0$ is called the density ratio. At 11 km $\sigma = 0.297$, so an airliner at that altitude needs to fly about $1/\sqrt{\sigma} = 1.83$ times faster than at sea level to generate the same lift at the same lift coefficient. Small model aircraft fly close to sea level, where $\sigma \approx 1$, but a hot day or a high-altitude field reduces density noticeably: at 1500 m in ISA conditions $\sigma = 0.86$.

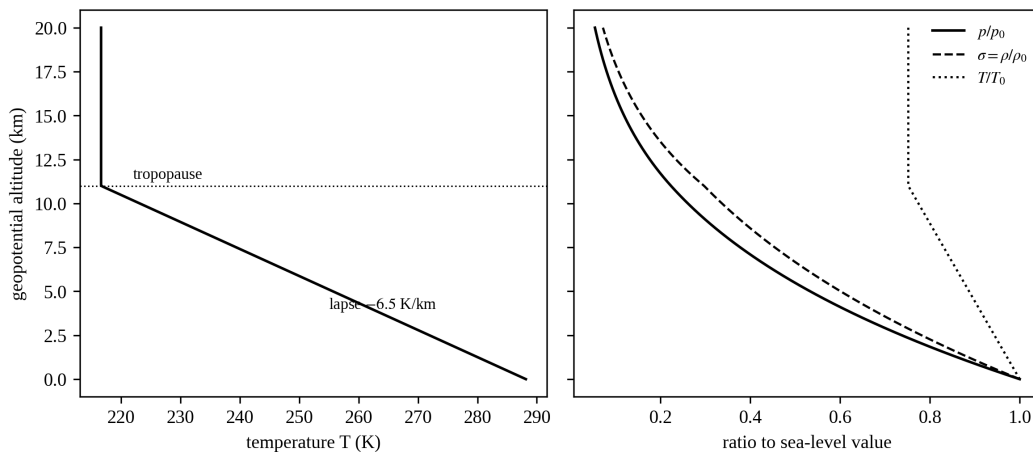


Figure 7. The International Standard Atmosphere. Temperature falls linearly up to the tropopause at 11 km and is constant above it. Pressure and density fall roughly exponentially.

2.5 Compressibility, the speed of sound and the Mach number

Pressure disturbances travel through air at the speed of sound,

$$a = \sqrt{\gamma RT}, \quad \gamma = 1.4$$

which is 340.3 m/s at sea level. The **Mach number** is the ratio of flight speed to the local speed of sound, $M = V/a$. When air is brought to rest isentropically its density rises according to

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/(\gamma - 1)} \approx 1 + \frac{M^2}{2}$$

so the fractional density change in the flow is of order $M^2/2$. At $M = 0.3$ this is about 4.5 percent, and below this Mach number the flow is conventionally treated as **incompressible**, with constant density. Every result in this module up to Section 15 assumes incompressible flow. A model flying wing at 20 m/s has $M \approx 0.06$ and is deeply incompressible. For moderate subsonic Mach numbers the

Prandtl–Glauert rule corrects incompressible pressure coefficients, $C_p = C_{p,0}/\sqrt{1 - M^2}$, and the same factor raises the lift-curve slope. Above the critical Mach number, where the local flow first reaches sonic speed, shock waves form and wave drag appears. Sweep (Section 15) delays this.

3. Fluid Kinematics

Kinematics describes motion without asking what causes it. Its tools, the velocity field, streamlines, the continuity equation, vorticity and circulation, are the vocabulary in which the theory of lift is written.

3.1 Describing a flow field and the material derivative

There are two ways to describe a moving fluid. In the **Lagrangian** description one follows individual fluid elements and records their properties as they move. In the **Eulerian** description one fixes attention on points in space and records the properties of whatever fluid passes through them, giving fields such as $\mathbf{V}(x, y, z, t)$. Aerodynamics almost always uses the Eulerian view, and most often in a frame of reference attached to the aircraft, in which the air streams past a stationary wing. The physical laws, however, apply to fluid elements. The rate of change of a property following a fluid element is the **material** (or substantial) **derivative**:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla$$

The first term is the **local** rate of change at a fixed point. The remaining terms form the **convective** rate of change, caused by the element moving to a place where the property is different. Even in a steady flow, where $\partial/\partial t = 0$, a fluid element accelerates as it moves around an airfoil, because the velocity field varies in space.

3.2 Streamlines, pathlines and steady flow

A **streamline** is a curve that is everywhere tangent to the velocity vector at a given instant. A **pathline** is the trajectory of a single fluid element. A **streakline** is the locus of all elements that have passed through a given point, which is what smoke from a fixed nozzle shows. In **steady** flow, where nothing depends on time, all three coincide. No fluid crosses a streamline, so a solid surface in steady flow is itself a streamline, and a bundle of streamlines forms a **stream tube** through which the mass flow is constant.

3.3 Conservation of mass: the continuity equation

Mass can be neither created nor destroyed. For a fixed control volume V bounded by a closed surface S , the rate of increase of mass inside equals the net rate of inflow across the surface:

$$\frac{\partial}{\partial t} \iiint_V \rho dV + \iint_S \rho \mathbf{V} \cdot \mathbf{n} dS = 0$$

Applying the divergence theorem to the surface integral and noting that the result must hold for any volume gives the differential form,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

For incompressible flow ρ is constant and this reduces to

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Applied to a stream tube with steady flow, the integral form gives $\rho AV = \text{constant}$, and for incompressible flow simply $A_1 V_1 = A_2 V_2$. Where streamlines crowd together the flow is faster. Over the upper surface of a lifting airfoil the streamlines are squeezed together, the flow accelerates, and by Bernoulli's equation the pressure falls.

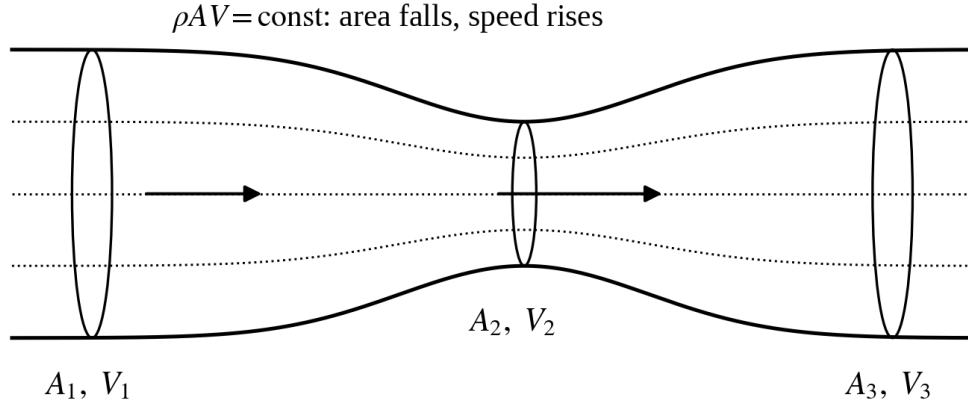


Figure 8. A stream tube. With no flow across its walls, the mass flow ρAV is the same at every section, so the velocity is highest where the area is smallest.

3.4 Rotation and vorticity

A fluid element can translate, rotate and deform. Its angular velocity is half the vorticity, $\omega = \nabla \times \mathbf{V}$. In the two-dimensional flows used for airfoils, with velocity components u along x and w along z (or u, v in an x - y plane), only the component normal to the plane exists, for example $\omega_z = \partial v / \partial x - \partial u / \partial y$. A flow in which $\omega = 0$ everywhere is **irrotational**. The flow outside the boundary layer and wake of a wing in uniform approaching flow is irrotational to an excellent approximation, because the oncoming air has no vorticity and, by Kelvin's theorem (Section 9.6), inviscid flow cannot create any. All the vorticity is confined to the thin viscous layers on the surface and in the wake.

3.5 Circulation

The **circulation** around a closed curve C is the line integral of the tangential velocity. In aerodynamics the sign is chosen so that circulation is positive when it is clockwise for a flow moving from left to right, since that is the sense that produces positive lift:

$$\Gamma \equiv - \oint_C \mathbf{V} \cdot d\mathbf{s}$$

where the integral is taken counter-clockwise. By Stokes's theorem,

$$\Gamma = - \iint_S (\nabla \times \mathbf{V}) \cdot \mathbf{n} dS$$

so the circulation equals the total vorticity enclosed, with a sign. Circulation is the quantity that links the flow to the force, through the Kutta-Joukowski theorem of Section 9.5, $L' = \rho_\infty V_\infty \Gamma$. Physically, a net circulation means that the air moves faster over the upper surface than over the lower surface, by an amount that integrates around the airfoil to Γ .

3.6 The velocity potential and Laplace's equation

Because the curl of a gradient is zero, any irrotational velocity field can be written as the gradient of a scalar, the **velocity potential** ϕ :

$$\mathbf{V} = \nabla \phi, \quad u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}$$

If the flow is also incompressible, substituting into $\nabla \cdot \mathbf{V} = 0$ gives Laplace's equation,

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

This is the governing equation of incompressible potential flow. It is **linear**, so the sum of any two solutions is also a solution. This principle of **superposition** is what makes potential-flow aerodynamics possible: complicated flows are built by adding simple ones (Section 9), and wings are modelled by adding many vortices (Section 14).

3.7 The stream function

In two-dimensional incompressible flow the continuity equation is satisfied automatically if the velocity is written in terms of a **stream function** ψ :

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Lines of constant ψ are streamlines, and the difference in ψ between two streamlines equals the volume flow rate per unit depth between them. For irrotational flow ψ also satisfies Laplace's equation, and lines of constant ϕ cross lines of constant ψ at right angles. Plots of streamlines in this module are contour plots of ψ .

4. Fluid Dynamics: Momentum and Energy

4.1 Forces on a fluid element

Newton's second law applied to a fluid element states that its mass times its acceleration equals the sum of the forces on it. Three kinds of force act: **pressure forces** on its faces, whose net value per unit volume is $-\nabla p$; **viscous forces** from the shear and normal stresses; and **body forces**, such as gravity, acting on its mass.

4.2 The momentum equation: Euler and Navier–Stokes

For an incompressible Newtonian fluid the momentum equation is the Navier–Stokes equation,

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{V} + \rho \mathbf{g}$$

Outside the boundary layer the viscous term is negligible, and dropping it gives the **Euler equation** for inviscid flow,

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{g}$$

The Euler equation says that a fluid element accelerates toward lower pressure. Around a curved streamline the pressure must therefore fall toward the centre of curvature, to provide the centripetal acceleration. This is the clearest physical statement of why a cambered airfoil produces lift: the flow is turned, and turning a flow requires a pressure difference across it.

The momentum equation can also be written for a control volume. For steady flow the net force on the fluid inside equals the net rate at which momentum leaves through the boundary:

$$\iint_S (\rho \mathbf{V} \cdot \mathbf{n}) \mathbf{V} dS = - \iint_S p \mathbf{n} dS + \mathbf{F}_{body}$$

This form is used to find the drag of an airfoil from measurements in its wake. With a wake-survey rake downstream, the momentum deficit gives the profile drag per unit span, $D' = \rho \int u_2 (V_\infty - u_2) dy$, where u_2 is the velocity across the wake, provided the static pressure there has recovered to the free-stream value.

4.3 Bernoulli's equation

Take the steady Euler equation without body forces and resolve it along a streamline, with s the distance along it. The acceleration along the streamline is $V dV/ds$, so

$$\rho V \frac{dV}{ds} = - \frac{dp}{ds} \Rightarrow dp = -\rho V dV$$

This is Euler's equation in differential form. For constant density it integrates directly to **Bernoulli's equation**:

$$p + \frac{1}{2} \rho V^2 = p_0 = \text{constant}$$

Its conditions must be remembered: the flow must be steady, inviscid and incompressible, and the result holds along a streamline. If the flow is also irrotational, the constant is the same on every streamline, so Bernoulli's equation holds between any two points of the flow. It does not hold inside a boundary layer or a wake, where viscous losses reduce the total pressure, nor across a propeller disc, where energy is added.

4.4 Static, dynamic and total pressure; airspeed measurement

The three terms of Bernoulli's equation have names. The **static pressure** p is the thermodynamic pressure. The **dynamic pressure** $q = \frac{1}{2} \rho V^2$ is the kinetic energy per unit volume, and it is the scale of every aerodynamic force: lift, drag and moment are all proportional to it. The **total or stagnation pressure** p_0 is the pressure reached if the flow is brought to rest without loss, as at the stagnation point on the nose of an airfoil.

A **pitot-static** probe measures airspeed by measuring the difference between total and static pressure. In incompressible flow,

$$V = \sqrt{\frac{2(p_0 - p_\infty)}{\rho}}$$

The instrument is calibrated with sea-level density, so it indicates the **equivalent airspeed** $V_E = V \sqrt{\sigma}$, the speed at sea level that gives the same dynamic pressure. Because aerodynamic forces depend on

dynamic pressure, an aircraft stalls at the same equivalent airspeed at any altitude. The true airspeed V is what determines range and the Reynolds number.

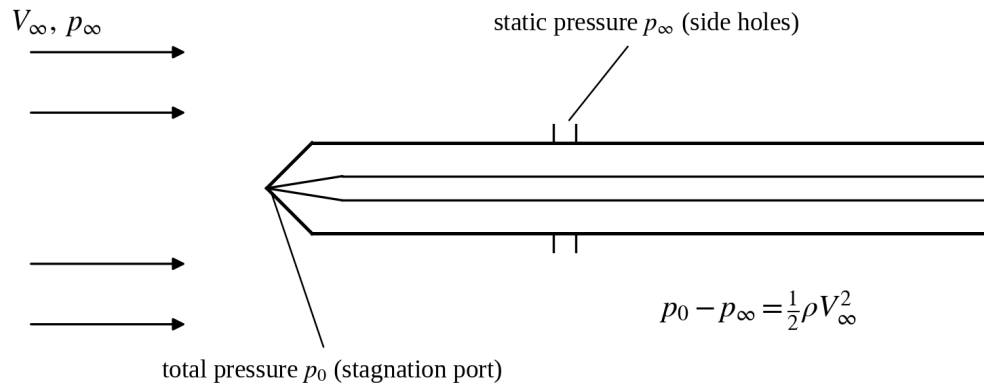


Figure 9. A pitot-static probe. The forward-facing hole senses total pressure and the side holes sense static pressure; their difference is the dynamic pressure.

4.5 The pressure coefficient

The **pressure coefficient** expresses the local pressure relative to the free stream, in units of the free-stream dynamic pressure:

$$C_p = \frac{p - p_\infty}{\frac{1}{2} \rho_\infty V_\infty^2} = 1 - \left(\frac{V}{V_\infty} \right)^2$$

The second form follows from Bernoulli's equation and so holds only in incompressible, irrotational flow. At a stagnation point $V = 0$ and $C_p = 1$, the largest possible value in incompressible flow. Where the flow is faster than the free stream, C_p is negative, which is called suction. Plots of C_p are conventionally drawn with negative values upward, so that the upper-surface curve lies above the lower-surface curve and the area between them is proportional to the lift. On a typical lifting airfoil most of the lift comes from suction on the forward part of the upper surface.

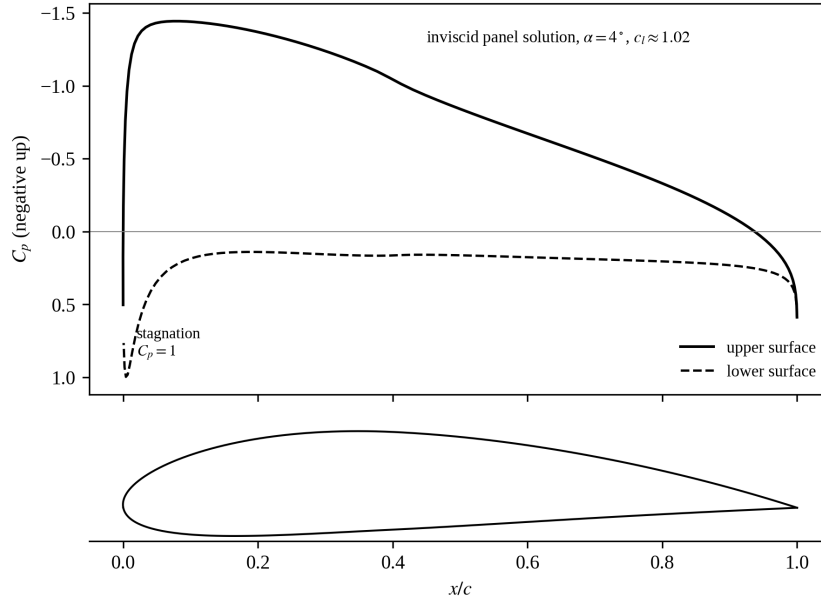


Figure 10. Pressure distribution on a NACA 4415 section at 4° angle of attack, computed with a Hess–Smith panel method. Suction near the upper-surface leading edge dominates, and the pressure recovers toward the trailing edge, which imposes an adverse pressure gradient on the boundary layer.

The rising pressure over the rear of the upper surface, called **pressure recovery**, is an **adverse pressure gradient** for the boundary layer. Section 6 shows that this gradient is what eventually causes separation and stall.

5. Dimensional Analysis and Similarity

5.1 The Buckingham Pi theorem

The lift on a wing depends on its size and shape, its attitude, and the properties and speed of the air: $L = f(\rho_\infty, V_\infty, l, \mu_\infty, \alpha, \text{shape})$. The **Buckingham Pi theorem** states that a relation among n physical quantities involving k independent dimensions (here mass, length and time) can be rewritten as a relation among $n - k$ dimensionless groups. The seven quantities above, less three dimensions, reduce to four groups, and the standard choice is

$$C_L = \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 S} = f(\alpha, Re, M, \text{shape})$$

This is a statement of enormous practical value. A lift coefficient measured on a model in a wind tunnel applies to a full-size aircraft of the same shape at the same angle of attack, provided the Reynolds and Mach numbers are also matched. The flow is then said to be **dynamically similar**.

5.2 The force and moment coefficients

Every aerodynamic force is divided by the dynamic pressure and a reference area, and every moment by the dynamic pressure, a reference area and a reference length. For a wing or an aircraft the reference area is the planform area S , and the reference length for pitching moment is the mean aerodynamic chord $\bar{c}$:

$$C_L = \frac{L}{q_\infty S}, \quad C_D = \frac{D}{q_\infty S}, \quad C_m = \frac{M}{q_\infty S \bar{c}}$$

For a two-dimensional section the quantities are per unit span, and the reference area per unit span is the chord c :

$$c_l = \frac{L'}{q_\infty c}, \quad c_d = \frac{D'}{q_\infty c}, \quad c_m = \frac{M'}{q_\infty c^2}$$

The coefficients are useful because, at fixed Reynolds and Mach numbers, they depend only on the angle of attack and the shape. A wing that produces $C_L = 0.5$ does so at any speed; the actual lift simply scales with $q_\infty S$.

5.3 The Reynolds number

The Reynolds number is the ratio of inertial to viscous forces in the flow,

$$Re = \frac{\rho_\infty V_\infty l}{\mu_\infty} = \frac{V_\infty l}{\nu_\infty}$$

with l a reference length, normally the chord. A high Reynolds number means that viscous effects are confined to thin boundary layers. A low one means that viscosity influences the whole flow more strongly. The Reynolds number controls whether the boundary layer is laminar or turbulent, how thick it is, the skin-friction coefficient, and the maximum lift coefficient.

Typical values span a wide range. An airliner wing with a 5 m chord at 230 m/s at cruise altitude ($\nu \approx 3.5 \times 10^{-5} \text{ m}^2/\text{s}$) has $Re \approx 3 \times 10^7$. A sailplane has about $1\text{--}2 \times 10^6$. The example flying wing of this module, with a mean aerodynamic chord of 0.233 m at 17 m/s near sea level, has $Re = 17 \times 0.233/1.461 \times 10^{-5} \approx 2.7 \times 10^5$, and its 0.15 m tip operates near 1.7×10^5 . This is the

low-Reynolds-number regime, in which airfoil behaviour is noticeably worse and more sensitive than in full-scale data, and it is why the flying-wing module stresses the low Reynolds number at the tips of a tapered wing.

5.4 The Mach number and other groups

The Mach number $M = V/a$ measures the importance of compressibility (Section 2.5). For the low-speed aircraft considered here it is small enough to ignore. Other groups exist, such as the reduced frequency for unsteady flow and the Froude number for problems involving gravity waves, but they do not enter the static design of a flying wing.

6. Viscous Flow and the Boundary Layer

In 1904 Prandtl resolved a paradox. Inviscid theory predicts lift well but predicts zero drag for any closed body (d'Alembert's paradox, Section 9.3). Prandtl showed that at high Reynolds number the effects of viscosity are confined to a thin **boundary layer** next to the surface. Outside it the flow is effectively inviscid and the theory of Sections 9, 10 and 14 applies. Inside it viscosity reduces the velocity to zero at the wall, producing skin friction, and the boundary layer's response to pressure gradients decides whether the flow stays attached or separates.

6.1 Boundary-layer thickness measures

The **boundary-layer thickness** δ is the distance from the wall at which the velocity reaches 99 percent of the outer (edge) velocity U_e . Two integral thicknesses are more useful. The **displacement thickness** δ^* is the distance by which the wall would have to be moved outward, in an inviscid flow, to produce the same mass-flow deficit; the outer flow sees the body as thickened by δ^* . The **momentum thickness** θ measures the momentum deficit and is directly related to drag:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_e}\right) dy, \quad \theta = \int_0^\delta \frac{u}{U_e} \left(1 - \frac{u}{U_e}\right) dy, \quad H = \frac{\delta^*}{\theta}$$

The **shape factor** H describes the fullness of the velocity profile. It is about 2.6 for a laminar layer with zero pressure gradient and about 1.3–1.4 for a turbulent one, and it rises as the layer approaches separation (roughly 3.5 for laminar and 2.4 for turbulent separation). The growth of θ along the surface is governed by von Kármán's momentum-integral equation,

$$\frac{d\theta}{dx} + (2 + H)\frac{\theta}{U_e} \frac{dU_e}{dx} = \frac{c_f}{2}$$

which shows explicitly how a decelerating outer flow ($dU_e/dx < 0$, an adverse pressure gradient) makes the momentum thickness grow faster.

6.2 Laminar and turbulent boundary layers

A **laminar** boundary layer flows in smooth layers, and momentum is exchanged between them only by molecular diffusion. A **turbulent** boundary layer contains chaotic eddies that mix fast fluid from the outer part of the layer down toward the wall. As a result the turbulent profile is much fuller near the wall, the wall shear stress is higher, and the layer is thicker and grows faster. For a flat plate with zero pressure gradient, with $Re_x = V_\infty x/\nu$, the classical results are

$$\begin{aligned} \text{laminar (Blasius): } \quad \delta/\bar{x} &= \frac{5.0}{\sqrt{Re_x}}, \quad c_f = \frac{0.664}{\sqrt{Re_x}}, \quad C_f = \frac{1.328}{\sqrt{Re_L}} \\ \text{turbulent: } \quad \delta/\bar{x} &= \frac{0.37}{Re_x^{1/5}}, \quad C_f = \frac{0.074}{Re_L^{1/5}} \end{aligned}$$

Here $c_f = \tau_w/q_\infty$ is the local skin-friction coefficient and C_f its average over a plate of length L . At $Re_L = 10^6$ the laminar value is 0.0013 and the turbulent value 0.0047, more than three times larger. This is why designers of low-drag airfoils try to keep the boundary layer laminar for as much of the chord as possible.

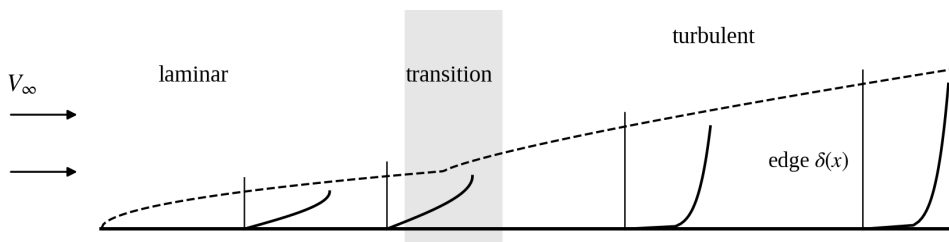


Figure 11. Development of the boundary layer along a flat plate. The laminar layer is thin with a gently curved profile; after transition the turbulent layer is thicker, grows faster, and has a fuller profile.

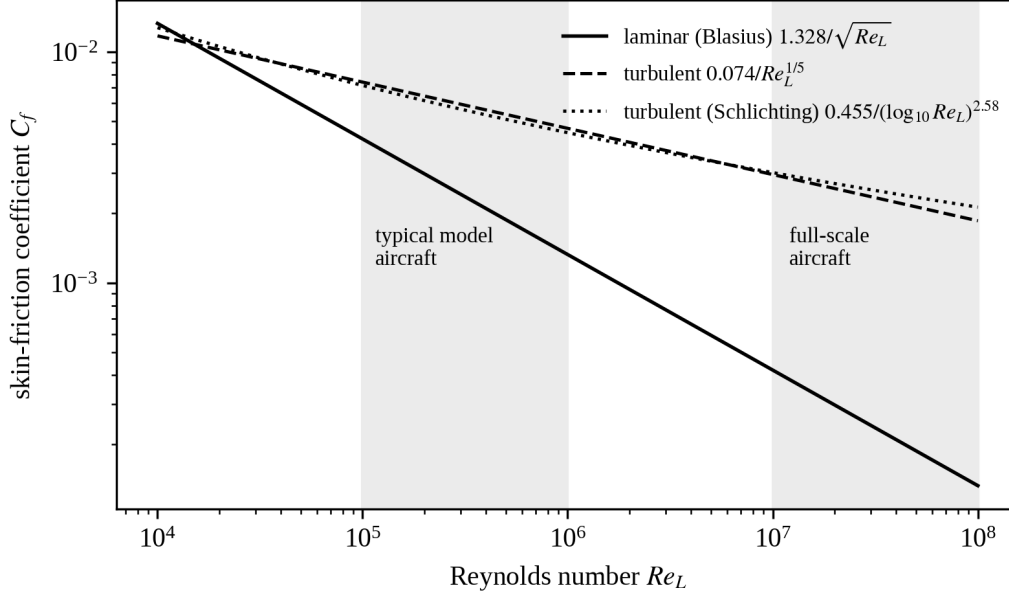


Figure 12. Average skin-friction coefficient of a flat plate. Turbulent friction is several times laminar friction. Model aircraft operate where the laminar–turbulent distinction is large and transition location matters most.

6.3 Transition

A laminar boundary layer becomes unstable and changes to turbulent flow through a process called **transition**. On a smooth flat plate in a quiet stream it begins at about $Re_x = 5 \times 10^5$, but the location depends strongly on the pressure gradient (a favourable gradient delays it, an adverse one hastens it), on surface roughness, on free-stream turbulence and on noise and vibration. Transition can be forced at a chosen location with a **trip strip**, a thin strip of roughness or a zigzag tape. The stall strips mentioned in the flying-wing module work by a related mechanism: a sharp edge placed at the leading edge of the root section forces the flow there to separate first at high angle of attack.

6.4 Pressure gradients and separation

Inside the boundary layer the pressure across the layer is essentially constant and equal to the outer-flow pressure, but the fluid near the wall has lost much of its momentum to friction. In a **favourable** pressure gradient ($dp/dx < 0$, accelerating outer flow) the pressure pushes the slow wall fluid forward and the layer is stable. In an **adverse** gradient ($dp/dx > 0$, decelerating outer flow) the pressure pushes it backward. If the adverse gradient is strong or long enough, the wall fluid is brought to rest and then reversed. The point where the wall shear stress falls to zero,

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$$

is the **separation point**. Downstream of it the boundary layer leaves the surface, a region of recirculating flow forms, and the outer flow no longer follows the body contour. A turbulent boundary layer, with its fuller profile and its continual supply of momentum from the outer layer, can withstand a much stronger adverse gradient than a laminar one before separating. This is the paradox of the golf ball: the dimples trip the boundary layer to turbulence, which delays separation and greatly reduces pressure drag.

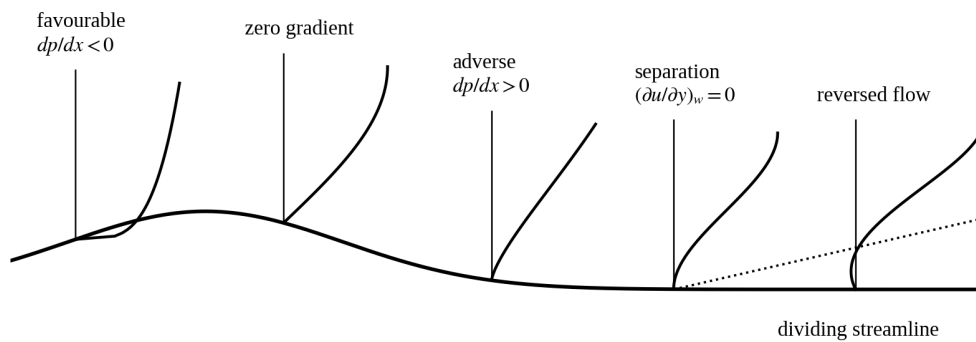


Figure 13. Velocity profiles through a region of increasing adverse pressure gradient. The profile becomes less full, the wall gradient falls to zero at separation, and reversed flow appears beyond it.

6.5 Separation and stall

On an airfoil at increasing angle of attack, the suction peak near the leading edge grows and the adverse pressure gradient over the upper surface becomes steeper. Eventually the boundary layer separates and the lift stops increasing: the airfoil **stalls**. Three types of stall are distinguished, depending mainly on the thickness and leading-edge radius. In **trailing-edge stall**, typical of thick sections, the separation point creeps forward gradually from the trailing edge, and the lift curve rounds over gently. In **leading-edge stall**, typical of moderately thin sections, a short separation bubble near the leading edge suddenly bursts and the flow separates from almost the whole upper surface at once, with an abrupt loss of lift. In **thin-airfoil stall**, typical of very thin sections, a long separation bubble grows steadily aft from the leading edge, the lift curve develops an early kink, and the maximum lift is low. Section 11 shows these on lift curves. Gentle trailing-edge stall is strongly preferred for the tip sections of a flying wing, where a sudden stall would drop a wing.

6.6 Low-Reynolds-number effects and the laminar separation bubble

Below a chord Reynolds number of about 5×10^5 the boundary layer usually remains laminar up to and beyond the point of minimum pressure. Being laminar, it cannot sustain the subsequent adverse gradient, and it separates. The separated shear layer is highly unstable, becomes turbulent within a short distance, and the increased mixing then lets it reattach to the surface as a turbulent boundary layer. The enclosed region of recirculating flow is a **laminar separation bubble**.

Bubbles cost drag and change the effective shape of the airfoil, and their size depends sensitively on the Reynolds number and angle of attack. At low enough Reynolds number the shear layer fails to reattach at all and the airfoil stalls prematurely. The practical consequences for a small flying wing are that the maximum lift coefficient is lower, drag is higher, and the polars of the same airfoil at the root and tip, which run at different Reynolds numbers, differ noticeably. Airfoil data must therefore always be taken at the Reynolds number at which each section will actually fly.

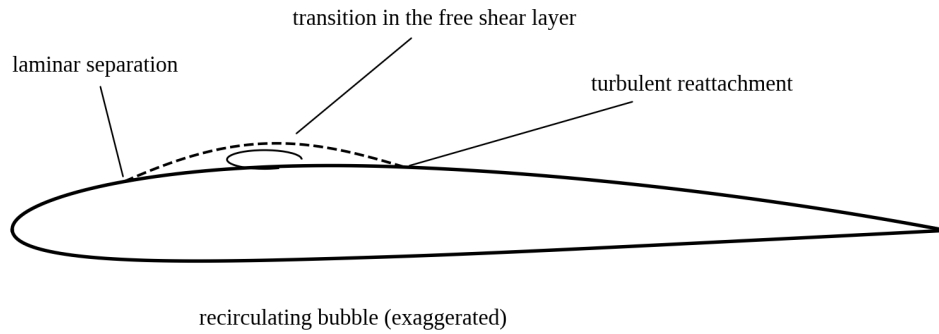


Figure 14. A laminar separation bubble on the upper surface of an airfoil at low Reynolds number (thickness exaggerated).

6.7 What the flying-wing module uses from this section

The boundary layer explains profile drag and the drag penalty of reflexed sections and geometric twist. Separation explains stall and why the tips, at lower Reynolds number, stall first on a tapered wing. Transition and trip strips explain the stall-strip remedy. The spanwise drift of the boundary layer on swept wings, which carries slow fluid toward the tips and promotes tip stall, is developed in Section 15.

7. Airfoil Geometry

7.1 Nomenclature

An **airfoil** (or wing section) is the cross-section of a wing cut parallel to the plane of symmetry. Its most forward point is the **leading edge** (LE) and its most rearward point the **trailing edge** (TE). The straight line joining them is the **chord line**, and its length is the **chord** c . The **mean camber line** is the locus of points midway between the upper and lower surfaces, measured perpendicular to the camber line itself. The maximum distance between the camber line and the chord line is the **maximum camber**, and its chordwise position is the position of maximum camber. The **thickness** is the distance between the surfaces measured perpendicular to the camber line; its maximum value, as a fraction of the chord, is the thickness ratio. The front of the section is rounded with a **leading-edge radius**, and the angle between the surfaces at the trailing edge is the **trailing-edge angle**.

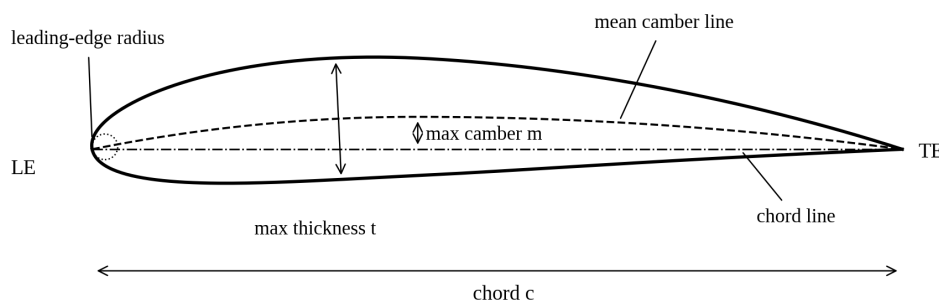


Figure 15. Airfoil nomenclature, illustrated on a NACA 4415 section (4 percent camber at 40 percent chord, 15 percent thickness).

Camber is what allows a section to produce lift at zero angle of attack. Thickness provides structural depth for the spar, volume for fuel or batteries, and a rounded nose that tolerates a range of angles of attack without leading-edge separation. Thin-airfoil theory (Section 10) shows that, to first order, lift and moment depend only on the camber line and the angle of attack, and thickness affects mainly the pressure distribution, drag and stall.

7.2 Constructing an airfoil from camber and thickness

An airfoil is generated by wrapping a symmetric thickness distribution $y_t(x)$ around a camber line $y_c(x)$. With $\theta = \arctan(dy_c/dx)$ the local slope of the camber line, the upper and lower surface coordinates are

$$x_u = x - y_t \sin \theta, \quad y_u = y_c + y_t \cos \theta, \quad x_l = x + y_t \sin \theta, \quad y_l = y_c - y_t \cos \theta$$

7.3 The NACA four-digit series

The NACA four-digit series is the simplest systematic family and is still widely used. In the designation NACA *MPTT*, the first digit is the maximum camber m in percent of chord, the second is the position of maximum camber p in tenths of chord, and the last two give the thickness t in percent of chord. NACA 4415 therefore has 4 percent camber at 40 percent chord and is 15 percent thick, and NACA 0012 is a symmetric section 12 percent thick. With x as a fraction of chord, the thickness distribution is

$$y_t = 5t(0.2969\sqrt{x} - 0.1260x - 0.3516x^2 + 0.2843x^3 - 0.1015x^4)$$

(the last coefficient is changed to -0.1036 for a closed, sharp trailing edge), and the camber line consists of two parabolic arcs meeting at the point of maximum camber:

$$y_c = \frac{m}{p^2}(2px - x^2) \quad (0 \leq x \leq p), \quad y_c = \frac{m}{(1-p)^2}[(1-2p) + 2px - x^2] \quad (p \leq x \leq 1)$$

7.4 Other families and the reflexed section

The NACA five-digit series places maximum camber further forward and was designed for low pitching moment. The NACA 6-series sections are shaped to keep the boundary layer laminar over a long run of favourable pressure gradient, which gives a "drag bucket" of low drag over a range of lift coefficients. Modern sections for sailplanes and model aircraft, by designers such as Eppler, Wortmann, Selig and Hepperle, are designed numerically for a specific Reynolds number range.

A **reflexed** section has an S-shaped camber line: positive camber over the front, and a trailing edge that curves back upward. Sections designed this way for tailless aircraft include the Eppler E 186 and the Hepperle MH 45, MH 60 and MH 61. Section 10.6 shows from thin-airfoil theory why the upturned trailing edge X that appear throughout the flying-wing module.

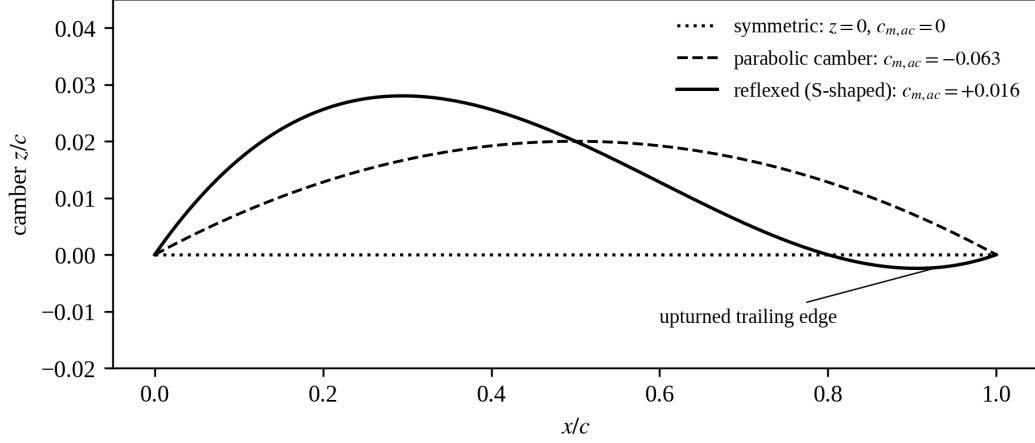


Figure 16. Three camber lines. The symmetric section has no camber and zero moment. The parabolic camber line gives a nose-down moment. The S-shaped reflexed line, whose trailing edge turns upward, gives a nose-up moment. The moment values are those derived in Section 10.

8. Aerodynamic Forces and Moments

8.1 The origin of aerodynamic force

Air can act on a body in only two ways: through the pressure p , acting normal to the surface, and through the shear stress τ , acting tangential to it. Every aerodynamic force and moment, however complicated the flow, is the integral of these two distributions over the surface. For a two-dimensional airfoil, with ds an element of surface length, the normal force N' perpendicular to the chord, the axial force A' along it, and the moment about the leading edge M'_{LE} are

$$N' = -\int_{LE}^{TE} (p_u \cos \theta + \tau_u \sin \theta) ds_u + \int_{LE}^{TE} (p_l \cos \theta - \tau_l \sin \theta) ds_l$$

$$A' = \int_{LE}^{TE} (-p_u \sin \theta + \tau_u \cos \theta) ds_u + \int_{LE}^{TE} (p_l \sin \theta + \tau_l \cos \theta) ds_l$$

$$M'_{LE} = \int_{LE}^{TE} [(p_u \cos \theta + \tau_u \sin \theta)x - (p_u \sin \theta - \tau_u \cos \theta)y] ds_u + \int_{LE}^{TE} [(-p_l \cos \theta + \tau_l \sin \theta)x + (p_l \sin \theta + \tau_l \cos \theta)y] ds_l$$

where θ is the local surface inclination, and the subscripts u and l denote the upper and lower surfaces. These expressions use the nose-up-positive convention of this module. In non-dimensional form, neglecting shear and noting that $dx = ds \cos \theta$, the normal-force coefficient is simply the area between the two C_p curves:

$$c_n = \frac{1}{c} \int_0^c (C_{p,l} - C_{p,u}) dx$$

This is why pressure plots are drawn with negative C_p upward: the enclosed area is the lift. Pressure contributes almost all of the lift and, before stall, a modest part of the drag. Shear contributes almost none of the lift and, before stall, most of the drag of a streamlined section.

8.2 Body axes and wind axes

The resultant force R' can be resolved in two ways. Relative to the chord it has components N' and A' . Relative to the free stream it has the **lift** L' , perpendicular to V_∞ , and the **drag** D' , parallel to it. The two pairs are related by a rotation through the angle of attack α :

$$L' = N' \cos \alpha - A' \sin \alpha, \quad D' = N' \sin \alpha + A' \cos \alpha$$

Lift is defined as perpendicular to the free stream, not to the chord and not to the ground. In a climb or a turn the lift is not vertical. At small angles of attack $L' \approx N'$, which allows the moment transfer formulas below to be written with the lift. Note also that the axial force on a lifting airfoil at moderate α is usually small and points forward, because the low pressure acting on the forward-facing nose pulls the section forward. This leading-edge suction is what cancels most of the rearward component of the normal force and keeps the drag low.

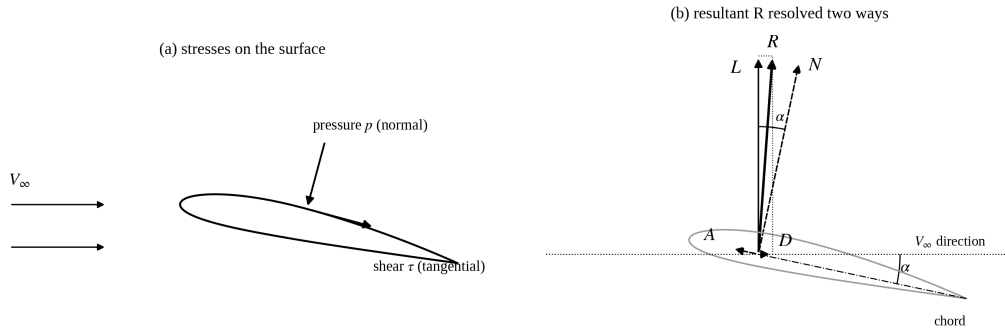


Figure 17. (a) Pressure acts normal to the surface and shear stress tangential to it. (b) The resultant R resolved into lift and drag (wind axes) or into normal and axial force (chord axes), which differ by a rotation through α .

8.3 Moments and the transfer of moments

A distributed force produces a moment whose value depends on the reference point. Take a reference point at chordwise position x_r . Each element of normal force dN' at position x produces a moment $-(x - x_r)dN'$ about it, the minus sign arising because an upward force aft of the reference point is nose-down. Integrating,

$$M'_r = -\int (x - x_r) dN' = M'_{LE} + x_r N'$$

In coefficient form, with $h = x/c$ and $N' \approx L'$, the moment about any point can be found from the moment about any other point:

$$c_{m,2} = c_{m,1} + (h_2 - h_1)c_l$$

This single relation, applied with point 1 at the aerodynamic centre and point 2 at the centre of gravity, is the starting equation of the longitudinal stability analysis in the flying-wing module: $C_{m,cg} = C_{m,ac} + C_L(h_{cg} - h_{ac})$. Its physical content is the lever rule: moving the reference point aft by a fraction Δh of the chord adds a nose-up moment equal to the lift times the extra lever arm.

8.4 The centre of pressure

The **centre of pressure** (CP) is the point about which the moment of the aerodynamic force is zero. It is where the resultant force can be considered to act. Setting $c_m = 0$ in the transfer formula with point 1 at the leading edge gives

$$h_{cp} = \frac{x_{cp}}{c} = -\frac{c_{m,LE}}{c_l}$$

Because both the moment and the lift change with angle of attack, the centre of pressure moves. As the lift tends to zero its position tends to infinity, which makes the CP an awkward reference point for

analysis even though it is easy to visualise.

8.5 The aerodynamic centre

Is there a point about which the moment does not change with angle of attack? Differentiate the transfer formula, with point 1 at the leading edge, and set the result to zero:

$$\frac{dc_{m,ac}}{d\alpha} = \frac{dc_{m,LE}}{d\alpha} + h_{ac} \frac{dc_l}{d\alpha} = 0 \Rightarrow h_{ac} = -\frac{dc_{m,LE}/d\alpha}{dc_l/d\alpha}$$

Because both c_l and $c_{m,LE}$ are linear in α over the useful range, this ratio is a constant, so such a point exists and its position is fixed. It is the **aerodynamic centre (AC)**. The moment about it, $c_{m,ac}$, is constant over the linear range, and the aerodynamic force system can be represented exactly by the lift acting at the AC plus this constant moment. Thin-airfoil theory (Section 10) shows that $c_{m,LE} = -c_l/4 + \text{constant}$, so $h_{ac} = 1/4$; the aerodynamic centre is at the quarter chord. Real sections have their AC within a few percent of this.

The relation between the CP and the AC follows from the transfer formula with point 1 at the AC and point 2 at the CP, where the moment is zero:

$$0 = c_{m,ac} + (h_{cp} - h_{ac})c_l \Rightarrow h_{cp} = h_{ac} - \frac{c_{m,ac}}{c_l}$$

This is the relation used in Section 2.2 of the flying-wing module. If $c_{m,ac}$ is negative (ordinary positive camber), the CP lies behind the AC and moves forward toward it as lift increases. If it is zero (symmetric section), the CP stays at the AC. If it is positive (reflexed section), the CP lies ahead of the AC and moves aft toward it as lift increases.

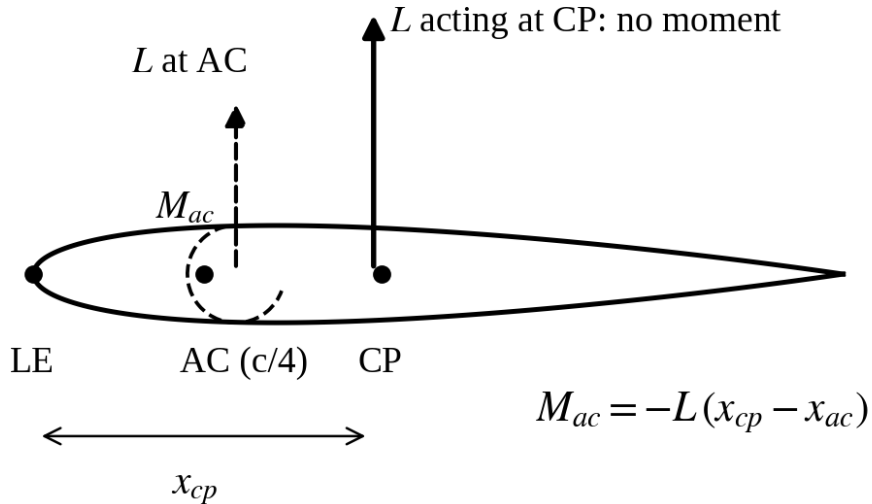


Figure 18. The same force system represented in two ways: the lift acting at the centre of pressure with no moment, or the lift acting at the aerodynamic centre together with the constant moment about it.

Worked example. A section has $c_{m,ac} = -0.06$ with the AC at the quarter chord. At $c_l = 0.3$ the centre of pressure is at $h_{cp} = 0.25 + 0.06/0.3 = 0.45$. At $c_l = 1.2$ it is at $0.25 + 0.05 = 0.30$. The moment about a point at 30 percent chord, for example a centre of gravity, at $c_l = 0.3$ is $c_m = -0.06 + (0.30 - 0.25)(0.3) = -0.045$, nose-down.

8.6 What the flying-wing module uses from this section

The resolution of the resultant into lift and drag, the transfer of moments along the chord, and the precise definitions of the centre of pressure and aerodynamic centre are used directly in its Sections 2 and 3. The rotation of the lift vector by a change in the local flow direction, introduced here, reappears as induced drag and as the induced thrust of the bell-shaped distribution.

9. Potential Flow and the Origin of Lift

Because Laplace's equation is linear, complicated incompressible irrotational flows can be built by adding simple ones. Four elementary flows are enough to explain the lift of a cylinder, and from there the lift of an airfoil.

9.1 Elementary flows

In polar coordinates (r, θ) about the origin, with V_r and V_θ the radial and tangential velocity components, the elementary flows are as follows. A **uniform flow** of speed V_∞ along x has

$$\phi = V_\infty r \cos \theta, \quad \psi = V_\infty r \sin \theta$$

A **source** of strength Λ (volume flow per unit depth) emits fluid radially, and a sink is a source of negative strength:

$$\phi = \frac{\Lambda}{2\pi} \ln r, \quad \psi = \frac{\Lambda}{2\pi} \theta, \quad V_r = \frac{\Lambda}{2\pi r}$$

A **doublet** of strength κ is the limit of a source and sink brought together while their strength increases:

$$\phi = \frac{\kappa}{2\pi} \frac{\cos \theta}{r}, \quad \psi = -\frac{\kappa}{2\pi} \frac{\sin \theta}{r}$$

A **point vortex** of strength Γ produces purely circular streamlines, with the tangential velocity inversely proportional to the radius. With the aerodynamic sign convention a positive Γ rotates clockwise:

$$\phi = -\frac{\Gamma}{2\pi} \theta, \quad \psi = \frac{\Gamma}{2\pi} \ln r, \quad V_\theta = -\frac{\Gamma}{2\pi r}$$

The point vortex is irrotational everywhere except at its centre, where the vorticity is concentrated. Any contour enclosing the centre has circulation Γ ; any contour not enclosing it has zero circulation. The point vortex is the single most important element in this module: thin airfoils are modelled as sheets of point vortices, and finite wings as systems of vortex lines.

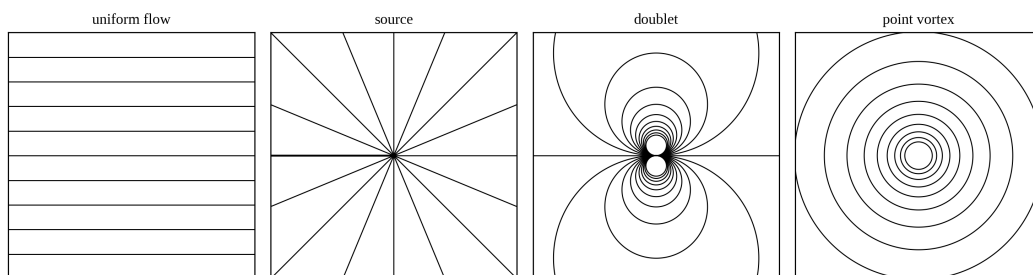


Figure 19. Streamlines of the four elementary flows. Each is a solution of Laplace's equation, and so is any sum of them.

9.2 Flow over a circular cylinder

Adding a uniform flow and a doublet with $\kappa = 2\pi V_\infty R^2$ produces a streamline that is a circle of radius R . Since a solid surface is a streamline, this is the flow around a circular cylinder:

$$V_r = \left(1 - \frac{R^2}{r^2}\right) V_\infty \cos \theta, \quad V_\theta = -\left(1 + \frac{R^2}{r^2}\right) V_\infty \sin \theta$$

On the surface $r = R$ the radial velocity is zero and $V_\theta = -2V_\infty \sin \theta$. The surface pressure coefficient is therefore

$$C_p = 1 - 4\sin^2 \theta$$

9.3 d'Alembert's paradox

The pressure distribution $1 - 4\sin^2 \theta$ is symmetric fore and aft and top and bottom, so its integral gives zero lift and zero drag. This result holds for any closed two-dimensional body in steady inviscid potential flow without circulation, and was known as d'Alembert's paradox, because real bodies plainly experience drag. The resolution is viscosity: the boundary layer separates on the rear of the cylinder, the pressure there never recovers, and pressure drag results. For streamlined bodies at small angles of attack the inviscid pressure field is nonetheless an excellent approximation, which is why potential flow predicts lift so well even though it cannot predict drag.

9.4 The lifting cylinder

Adding a point vortex of strength Γ at the centre leaves the circle a streamline, because the vortex induces only tangential velocity. The surface velocity becomes

$$V_\theta = -2V_\infty \sin \theta - \frac{\Gamma}{2\pi R}$$

The flow over the top is now faster than the flow underneath. The stagnation points move down to where $\sin \theta = -\Gamma/(4\pi V_\infty R)$, and the pressure distribution loses its top-to-bottom symmetry. Integrating the pressure gives a force perpendicular to the free stream and no drag. This is the **Magnus effect** seen on a spinning ball, where the circulation is created by the spin.

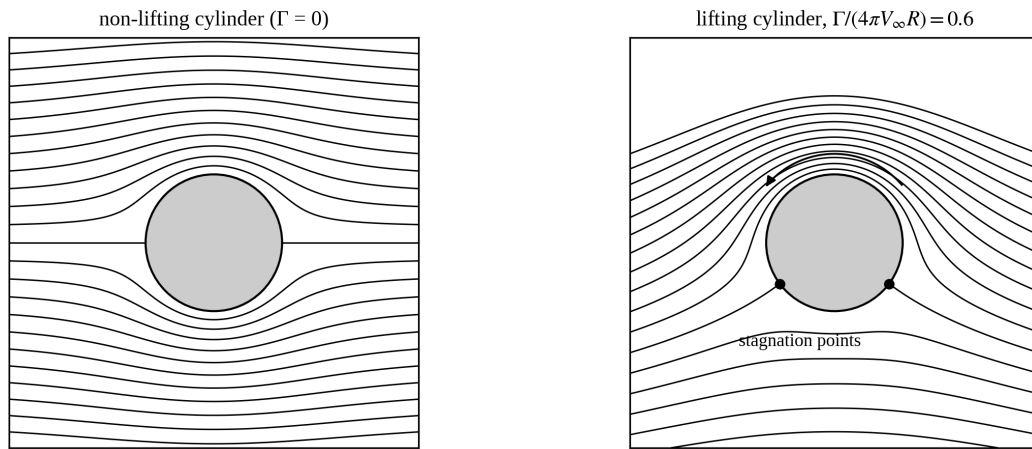


Figure 20. Flow over a cylinder without circulation (left) and with clockwise circulation (right). The circulation speeds the flow over the top, moves both stagnation points downward, and produces lift.

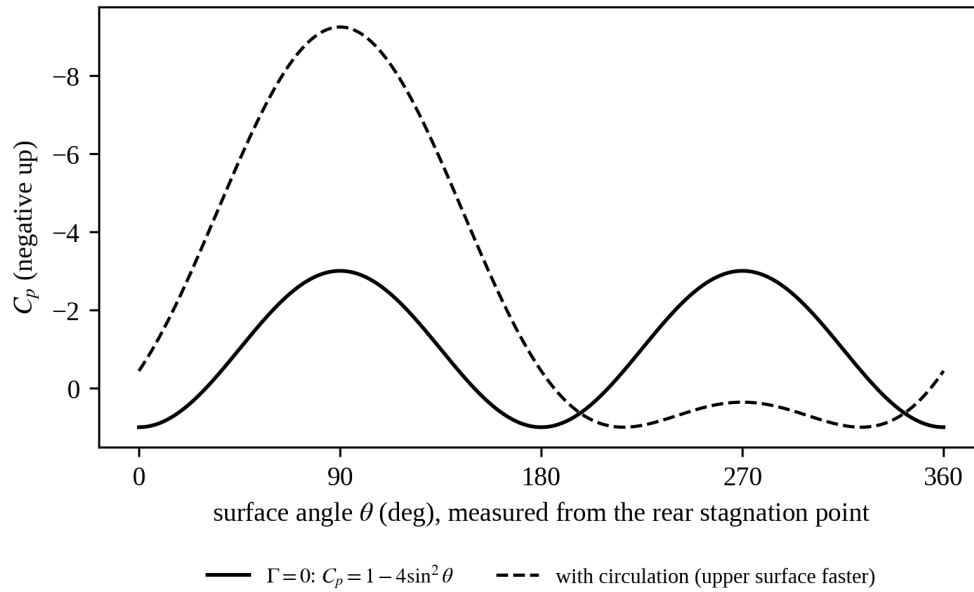


Figure 21. Surface pressure coefficient around the cylinder. Without circulation the distribution is symmetric and gives no net force; with circulation the upper surface carries much stronger suction.

9.5 The Kutta–Joukowski theorem

Integrating the surface pressure of the lifting cylinder gives a lift per unit span $L' = \rho_{\infty} V_{\infty} \Gamma$.

Remarkably, the result is not special to the cylinder. By applying the momentum equation to a large control volume around any two-dimensional body in steady incompressible potential flow, one finds that the lift depends only on the circulation around the body:

$$L' = \rho_{\infty} V_{\infty} \Gamma$$

This is the **Kutta–Joukowski theorem**, and it reduces the problem of predicting lift to the problem of predicting circulation. The theorem also gives a direction: the force is perpendicular to the free stream, so in inviscid two-dimensional flow there is no drag. Applied locally to each section of a finite wing, where the local flow is tilted by the downwash, the same theorem produces a force perpendicular to the *local* flow, and its rearward component is the induced drag (Section 14).

9.6 The Kutta condition, Kelvin's theorem and the starting vortex

Potential theory alone permits any value of the circulation around an airfoil: every value of Γ gives a mathematically valid flow. In most of them the flow from the lower surface whips around the sharp trailing edge at infinite velocity, which a real viscous fluid cannot do. Nature selects the one value of circulation for which the flow leaves the trailing edge smoothly. This is the **Kutta condition**: for a sharp trailing edge, the flow leaves the trailing edge smoothly and the velocity there is finite. In terms of a vortex sheet it requires the sheet strength to vanish at the trailing edge, $\gamma(\text{TE}) = 0$. The Kutta condition is a model of the effect of viscosity, imposed on an inviscid theory, and it is what makes the predicted lift unique and correct.

Kelvin's circulation theorem states that in an inviscid flow the circulation around a closed curve that always consists of the same fluid elements does not change with time. Consider an airfoil starting from rest. Initially the circulation around a large contour enclosing the airfoil is zero. As the airfoil accelerates, the flow around the trailing edge rolls up into a vortex, the **starting vortex**, which is shed into the wake. Since the circulation around the large contour must remain zero, an equal and opposite

circulation, the **bound vortex**, is left around the airfoil. The starting vortex can be photographed, and its existence confirms that the circulation around a lifting airfoil is real and not merely a mathematical device.

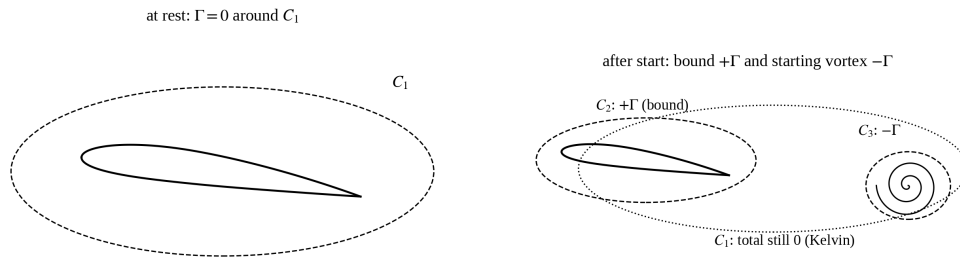


Figure 22. Kelvin's theorem and the starting vortex. When the airfoil starts, a vortex of circulation $-\Gamma$ is shed from the trailing edge, leaving the bound circulation $+\Gamma$ around the airfoil so that the total around the large contour remains zero.

9.7 A physical explanation of lift

Several popular explanations of lift are wrong. The "equal transit time" explanation, which claims that air parting at the leading edge must rejoin at the trailing edge, is false: the upper-surface air in fact arrives well before the lower-surface air. The correct explanation is consistent from several viewpoints. The airfoil shape and angle of attack, together with the Kutta condition, fix a circulation. The circulation means faster flow above and slower flow below, and by Bernoulli's equation lower pressure above and higher pressure below; integrated over the surface, this pressure difference is the lift. Equivalently, the airfoil turns the flow downward, giving the air downward momentum, and by Newton's third law the air pushes the airfoil up. The pressure and momentum descriptions are two views of the same physics, not competing theories, and both are contained in the Euler equation.

10. Thin-Airfoil Theory

Thin-airfoil theory, developed by Munk, Glauert and others in the 1920s, predicts the lift and moment of an airfoil from its camber line alone. It is the source of three results quoted constantly in the flying-wing module: the lift-curve slope of 2π per radian, the location of the aerodynamic centre at the quarter chord, and the dependence of the zero-lift moment on the shape of the camber line, which is what makes a reflexed section work.

10.1 Assumptions and model

The airfoil is assumed thin, with small camber and at a small angle of attack, so that the thickness can be ignored and the camber line can be treated as lying almost on the chord. The airfoil is replaced by a **vortex sheet** of strength $\gamma(x)$ per unit length placed along the camber line, or, to the same order of accuracy, along the chord. The total circulation is $\Gamma = \int \gamma dx$, and by Kutta–Joukowski the lift per unit span is $\rho_\infty V_\infty \Gamma$. The task is to find the $\gamma(x)$ that makes the camber line a streamline and satisfies the Kutta condition.

10.2 The fundamental equation

The camber line $z(x)$ is a streamline if the total velocity normal to it is zero. The free stream contributes a normal component $V_\infty (\alpha - dz/dx)$ for small angles. A vortex element $\gamma(\xi)d\xi$ at position ξ induces at point x on the chord a vertical velocity $\gamma(\xi)d\xi/[2\pi(x - \xi)]$. The sum of these must cancel the free-stream component, which gives the **fundamental equation of thin-airfoil theory**:

$$\frac{1}{2\pi} \int_0^c \frac{\gamma(\xi) d\xi}{x - \xi} = V_\infty \left(\alpha - \frac{dz}{dx} \right)$$

together with the Kutta condition $\gamma(c) = 0$. This is an integral equation for γ . It is linear, so the effects of angle of attack and of camber can be found separately and added.

10.3 The Glauert transformation

The substitution $x = (c/2)(1 - \cos \theta)$, with θ running from 0 at the leading edge to π at the trailing edge, is the key to the solution. The circulation is sought in the form

$$\gamma(\theta) = 2V_\infty \left[A_0 \frac{1 + \cos \theta}{\sin \theta} + \sum_{n=1}^{\infty} A_n \sin n\theta \right]$$

The first term is singular at the leading edge, which represents the infinite suction peak of a sharp-nosed thin plate, and every term vanishes at $\theta = \pi$, so the Kutta condition is satisfied automatically. Substituting into the fundamental equation and using the Glauert integral of Section 1.7 reduces it to

$$A_0 - \sum_{n=1}^{\infty} A_n \cos n\theta_0 = \alpha - \frac{dz}{dx}$$

This is a Fourier cosine series for the camber-line slope, and the coefficients follow from the orthogonality relations:

$$A_0 = \alpha - \frac{1}{\pi} \int_0^\pi \frac{dz}{dx} d\theta_0, \quad A_n = \frac{2}{\pi} \int_0^\pi \frac{dz}{dx} \cos n\theta_0 d\theta_0$$

Only A_0 depends on the angle of attack. All the other coefficients depend only on the shape of the camber line.

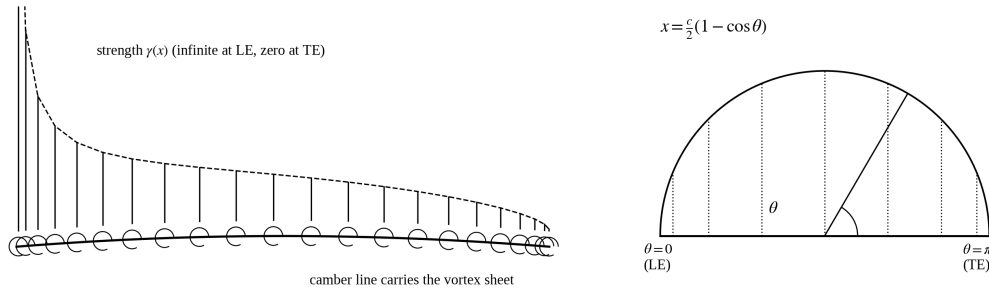


Figure 23. Left: the thin airfoil is replaced by a vortex sheet on its camber line, whose strength is infinite at the leading edge and zero at the trailing edge. Right: the Glauert variable θ , which clusters points toward both edges.

10.4 Results

Integrating the circulation gives the lift coefficient, and integrating the moment of each vortex element gives the moments:

$$c_l = \pi(2A_0 + A_1) = 2\pi(\alpha - \alpha_{L=0})$$

$$c_{m,LE} = -\frac{\pi}{2} \left(A_0 + A_1 - \frac{A_2}{2} \right) = -\frac{c_l}{4} + \frac{\pi}{4}(A_2 - A_1)$$

$$c_{m,c/4} = \frac{\pi}{4}(A_2 - A_1)$$

with the zero-lift angle of attack

$$\alpha_{L=0} = -\frac{1}{\pi} \int_0^\pi \frac{dz}{dx} (\cos \theta_0 - 1) d\theta_0$$

Four conclusions of permanent importance follow. First, the lift-curve slope is $dc_l/d\alpha = 2\pi$ per radian for every thin airfoil, whatever its camber; measured slopes are typically 5 to 10 percent less because of the boundary layer. Second, camber shifts the lift curve sideways, through $\alpha_{L=0}$, without changing its slope; positive camber makes $\alpha_{L=0}$ negative. Third, the moment about the quarter chord does not depend on α at all, so the quarter chord is the aerodynamic centre. Fourth, the moment about the aerodynamic centre depends only on the camber line, through the difference $A_2 - A_1$.

For a **symmetric** section $dz/dx = 0$, so $A_0 = \alpha$ and all other coefficients vanish. Then $c_l = 2\pi\alpha$, $c_{m,c/4} = 0$, and the centre of pressure is fixed at the quarter chord.

10.5 Worked example: a parabolic camber line

Take $z/c = 4\epsilon(x/c)(1 - x/c)$, a circular-arc-like camber line with maximum camber ϵ at mid-chord. Its slope is $dz/dx = 4\epsilon(1 - 2x/c)$, and since $1 - 2x/c = \cos \theta$ this is simply $4\epsilon \cos \theta$. Then $A_0 = \alpha$ (the integral of $\cos \theta$ over 0 to π is zero), $A_1 = 4\epsilon$ and $A_2 = 0$. Hence

$$c_l = 2\pi(\alpha + 2\epsilon), \quad \alpha_{L=0} = -2\epsilon, \quad c_{m,c/4} = -\pi\epsilon$$

For 2 percent camber ($\epsilon = 0.02$), $\alpha_{L=0} = -0.04$ rad = -2.3° and $c_{m,ac} = -0.063$. The moment is nose-down, as for every ordinary positively cambered section. For the NACA 4415 the same integrals, evaluated numerically for its two-arc camber line, give $\alpha_{L=0} = -4.15^\circ$ and $c_{m,c/4} = -0.106$.

10.6 Worked example: why reflex gives a nose-up moment

The moment formula $c_{m,ac} = (\pi/4)(A_2 - A_1)$ shows exactly what a designer must do to obtain a positive moment: make A_2 larger than A_1 . Consider a camber line whose slope is $dz/dx = B_0 + B_1 \cos \theta + B_2 \cos 2\theta$. Then $A_1 = B_1$, $A_2 = B_2$ and $A_0 = \alpha - B_0$. The $\cos 2\theta$ term, written in terms of x , is $2(1 - 2x/c)^2 - 1$, which is positive near both the leading and trailing edges and negative in the middle. Its contribution to the camber line therefore turns the trailing edge upward: it is the reflex. Integrating the slope, with $B_0 = B_2/3$ so that the camber line ends on the chord line, gives

$$\frac{z}{c} = B_0 x + B_1(x - x^2) + B_2 \left(x - 4x^2 + \frac{8}{3}x^3 \right), \quad \alpha_{L=0} = B_0 - \frac{B_1}{2}, \quad c_{m,ac} = \frac{\pi}{4}(B_2 - B_1)$$

With $B_1 = 0.08$, which alone is the parabolic camber line of the previous example with $\epsilon = 0.02$, and a reflex term $B_2 = 0.10$, the result is $\alpha_{L=0} = -0.38^\circ$ and $c_{m,ac} = +0.016$. Adding the reflex has changed the moment from -0.063 to $+0.016$ and moved the zero-lift angle from -2.3° to -0.4° , so at a given angle of attack the reflexed section produces less lift. This is precisely the trade described in Section 3.4 of the flying-wing module: positive moment is bought with lift. These are the camber lines plotted in Section 7.4.

The weighting functions in the formulas explain the result physically. The zero-lift angle weights the camber slope by $(\cos \theta - 1)$, which is largest near the trailing edge, so trailing-edge camber has the strongest effect on lift. The moment weights it through the $\cos \theta$ and $\cos 2\theta$ harmonics, so camber near the trailing edge, far from the quarter chord, produces large moments. Reflex is a small amount of negative camber placed where it has the greatest leverage.

10.7 Limits of thin-airfoil theory

Thin-airfoil theory ignores thickness and viscosity. It therefore predicts no drag and no stall, and it slightly overpredicts the lift-curve slope. Thickness, handled by panel methods such as XFOIL, moves the aerodynamic centre slightly and changes the pressure distribution, but the theory's predictions of slope, zero-lift angle and moment remain good guides for sections of up to about 12–15 percent thickness. The panel solution of Section 4.5 for NACA 4415 at 4° gives an inviscid lift coefficient of 1.02, about 14 percent higher than the thin-airfoil value of $2\pi(4^\circ + 4.15^\circ) = 0.89$, because in inviscid flow thickness increases the lift slope (roughly by a factor $1 + 0.77t/c$). In viscous flow the boundary layer removes most of this gain, which is why thin-airfoil theory agrees well with experiment for moderate thickness.

11. Airfoil Characteristics and Polars

Measured or computed airfoil data are presented as **polars**: plots of the coefficients against angle of attack and against each other, each for a stated Reynolds number. The flying-wing module asks the designer to "study the airfoil polars" to find zero-lift angles and moment coefficients, so this section explains how to read them.

11.1 The lift curve

The plot of c_l against α is the **lift curve**. Its features are the zero-lift angle $\alpha_{L=0}$, where it crosses the axis; the lift-curve slope a_0 in the linear range, close to 2π per radian (about 0.105–0.11 per degree); the departure from linearity as separation begins; and the **maximum lift coefficient** $c_{l,max}$ at the stall angle. In the linear range the curve is fully described by two numbers,

$$c_l = a_0(\alpha - \alpha_{L=0})$$

which is why, for twist calculations, the flying-wing module needs only the zero-lift angles of the root and tip sections.

11.2 The moment curve

The plot of c_m about the quarter chord against α is nearly flat over the linear range, which confirms that the quarter chord is close to the aerodynamic centre. Its level is $c_{m,ac}$: about -0.1 for a typical cambered section, zero for a symmetric one, and small and positive for a reflexed one. A slight slope in the measured curve means that the true aerodynamic centre is a little ahead of or behind the quarter chord, and its position can be found from the transfer formula of Section 8.5, $h_{ac} = 0.25 - (dc_{m,c/4}/d\alpha)/(dc_l/d\alpha)$.

11.3 The drag polar

The plot of c_l against c_d is the **drag polar**. For a two-dimensional section the drag is the **profile drag**, the sum of skin friction and the pressure drag caused by the boundary layer's displacement effect and any separation. The polar shows the minimum drag coefficient and the lift coefficient at which it occurs, the width of any low-drag bucket, and the rapid rise in drag approaching stall. A straight line from the origin tangent to the polar touches it at the point of maximum lift-to-drag ratio, since the slope of such a line is c_l/c_d .

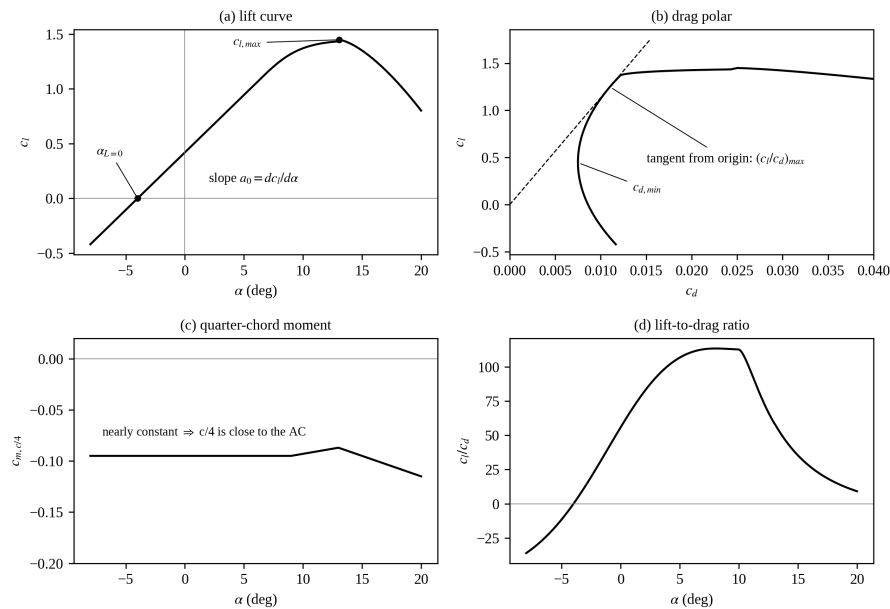


Figure 24. Illustrative polars of a cambered section similar to NACA 4415 at a Reynolds number near 10^6 . (a) Lift curve with its zero-lift angle, slope and maximum. (b) Drag polar with the tangent that marks the best lift-to-drag ratio. (c) Quarter-chord moment, nearly constant until stall. (d) Lift-to-drag ratio against angle of attack.

11.4 Stall types on the lift curve

The three stall mechanisms of Section 6.5 have characteristic lift curves. Trailing-edge stall gives a rounded peak and a gradual loss of lift beyond it. Leading-edge stall gives a sharp peak and a sudden drop. Thin-airfoil stall gives an early departure from linearity and a low, flat maximum. A wing whose tips have a gentle stall and whose root has an earlier or more abrupt one will stall from the root outward, which preserves roll control at the tips.

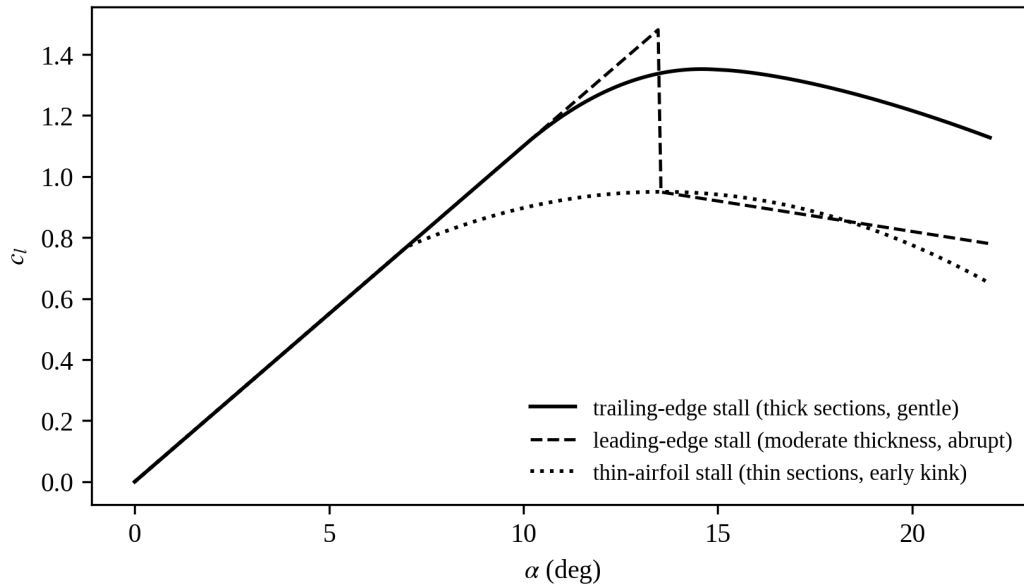


Figure 25. Characteristic lift curves for the three types of airfoil stall (illustrative).

11.5 The effect of Reynolds number

Lowering the Reynolds number raises the drag at every lift coefficient, because the skin-friction coefficient increases and separation bubbles appear. It lowers $c_{l,max}$ and the stall angle, because the boundary layer separates more readily. It often adds kinks to the polar where bubbles form and burst. On a tapered flying wing, whose tip chord may be half the root chord, the tip sections run at half the root Reynolds number and have both less lift capability and more drag than the same section at the root.

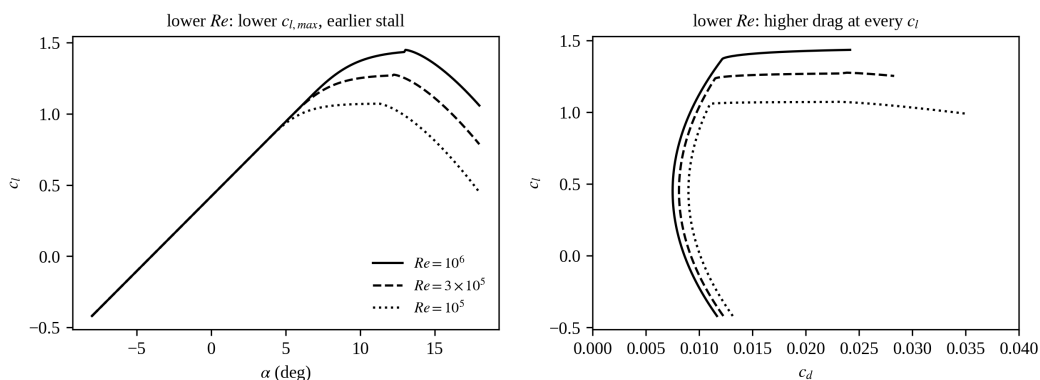


Figure 26. Illustrative effect of Reynolds number on the polars of the same section. Lower Reynolds number reduces the maximum lift and stall angle and increases drag.

11.6 Reading a polar in practice

Programs such as XFOIL and its graphical front end XFLR5 compute polars by coupling a panel method with a boundary-layer solution, and they are the usual source of section data for small aircraft. To extract the numbers a flying-wing designer needs, read the zero-lift angle where the lift curve crosses zero; the lift-curve slope from two points in the linear range; the moment coefficient about the quarter chord in the linear range; $c_{l,max}$ and the stall angle, preferably at the lowest Reynolds number the section will meet; and the minimum drag and the lift coefficient for best c_l/c_d . For each section, record the Reynolds number used, the transition criterion (often expressed as an N_{crit} value) and whether transition was free or forced, since all of these change the result.

12. Drag and Performance of the Complete Aircraft

12.1 The components of drag

The drag of a complete aircraft has several components, and naming them precisely avoids confusion. **Skin-friction drag** is the integral of the wall shear stress. **Pressure drag** (or form drag) is the streamwise component of the pressure force caused by the boundary layer's displacement and by separation; together these two make up the **profile drag** of a section. **Interference drag** is the extra drag that arises where components meet, such as at a wing–fuselage or wing–fin junction, because the boundary layers and pressure fields interact. **Induced drag** (drag due to lift) is the rearward tilt of the lift on a finite wing, caused by the downwash of the trailing vortices; it exists even in inviscid flow and is the subject of Section 14. **Wave drag** arises from shock waves at transonic and supersonic speed. The sum of all drag not due to lift is the **parasite drag**.

A flying wing gains its efficiency by eliminating the fuselage and tail, which removes their friction and form drag, the interference drag at their junctions, and much of the wetted area. It may lose some of that gain in **trim drag**, the extra drag incurred because the wing must produce a pitching moment to trim, through reflex or twist, rather than flying at its best spanwise loading.

12.2 Estimating parasite drag

A simple and useful estimate of the zero-lift drag coefficient uses the **wetted area** S_{wet} , the total area exposed to the flow, and an equivalent skin-friction coefficient C_{fe} that includes typical form and interference effects:

$$C_{D0} = C_{fe} \frac{S_{wet}}{S}$$

For a thin wing $S_{wet} \approx 2.04S$. Typical values of C_{fe} are 0.003 for a clean jet airliner and 0.005–0.006 for a small, less refined aircraft. A pure flying wing has the smallest possible ratio S_{wet}/S , which is the quantitative content of the claim that it is the lowest-drag configuration.

12.3 The parabolic drag polar

For an aircraft in the attached-flow range the drag coefficient is well represented by

$$C_D = C_{D0} + KC_L^2, \quad K = \frac{1}{\pi A R e}$$

where e is the **Oswald efficiency factor**, typically 0.75–0.9 for a complete aircraft. It includes the span efficiency of the lift distribution (Section 14.7) and the increase of profile drag with lift. The lift-to-drag ratio is greatest where the parasite and induced drag are equal:

$$C_L^* = \sqrt{\frac{C_{D0}}{K}}, \quad \left(\frac{L}{D}\right)_{max} = \frac{1}{2\sqrt{KC_{D0}}}$$

The speed at which this occurs in level flight, the minimum-drag speed, follows from $L = W$:

$$V_{md} = \sqrt{\frac{2W}{\rho S C_L^*}}$$

The power required is DV , and it is least at a lower speed, where $C_L = \sqrt{(3C_{D0}/K)}$ and the induced drag is three times the parasite drag. Minimum power gives the longest endurance for a propeller or electric aircraft and the minimum sink rate for a glider; maximum L/D gives the flattest glide and, for a propeller aircraft, the longest range.

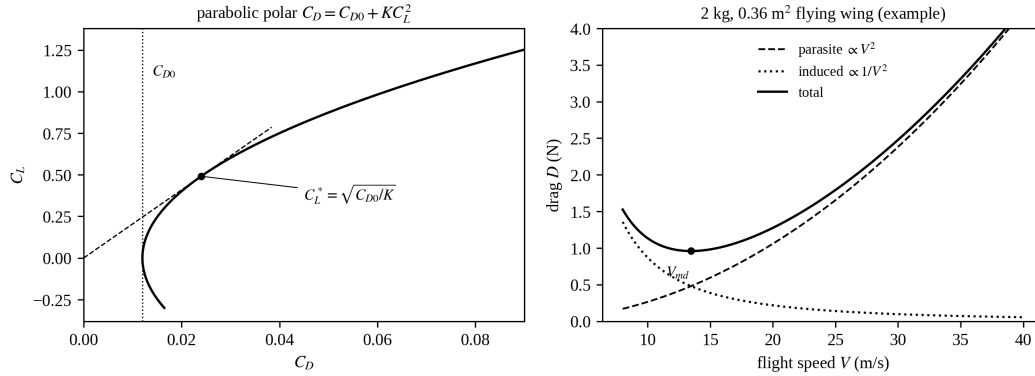


Figure 27. Left: the parabolic drag polar and the tangent that defines maximum L/D . Right: drag against speed for the example flying wing; induced drag falls and parasite drag rises with speed, and the total is least at the minimum-drag speed.

12.4 Worked example: the example flying wing

The example aircraft used throughout this module is a small electric flying wing of mass 2.0 kg ($W = 19.6$ N), span 1.6 m and area 0.36 m^2 , so $AR = 7.11$ and the wing loading is $W/S = 54.5 \text{ N/m}^2$. Take $C_{D0} = 0.012$ and $e = 0.9$. Then $K = 1/(\pi \times 7.11 \times 0.9) = 0.0498$, $C_L^* = \sqrt{(0.012/0.0498)} = 0.49$, and $(L/D)_{\max} = 1/(2\sqrt{(0.0498 \times 0.012)}) = 20.5$. At sea level the minimum-drag speed is $V_{md} = \sqrt{(2 \times 19.6/(1.225 \times 0.36 \times 0.49))} = 13.5 \text{ m/s}$. The minimum-power condition is at $C_L = 0.85$, where $L/D = 17.7$. In a glide at $(L/D)_{\max}$ the aircraft covers 20.5 m horizontally for every metre of height lost.

12.5 Range and endurance

The **Breguet range equation** follows from integrating the rate of fuel burn over the flight, with the aircraft held at constant lift-to-drag ratio. For a jet with thrust-specific fuel consumption c_T (fuel weight flow per unit thrust) and for a propeller aircraft with power-specific fuel consumption c and propeller efficiency η_p ,

$$R_{jet} = \frac{V}{c_T} \frac{L}{D} \ln \frac{W_0}{W_1}, \quad R_{prop} = \frac{\eta_p}{c} \frac{L}{D} \ln \frac{W_0}{W_1}$$

where W_0 and W_1 are the initial and final weights. A battery-electric aircraft does not lose weight, and its range is simply the usable energy E times the overall efficiency η from battery to thrust power, divided by the drag:

$$R_{elec} = \frac{\eta E}{D} = \eta E \frac{L/D}{W}$$

For the example aircraft with a 0.45 kg battery at 180 Wh/kg (81 Wh, or 2.9×10^5 J), 80 percent usable, $\eta = 0.45$ and $L/D = 20.5$, the ideal range is $0.45 \times 0.8 \times 2.9 \times 10^5 \times 20.5/19.6 \approx 110 \text{ km}$. In every one of these equations the range is proportional to L/D , which is why the lower drag of the flying wing translates directly into range.

12.6 Gliding flight

In a steady glide with no thrust, the lift, drag and weight balance. Resolving along and perpendicular to the flight path, inclined at angle γ below the horizon, gives $D = W \sin \gamma$ and $L = W \cos \gamma$, so

$$\tan \gamma = \frac{D}{L} = \frac{1}{L/D}, \quad w_{\text{sink}} = V \sin \gamma$$

The flattest glide is at maximum L/D , and the minimum sink rate is at maximum $C_L^{3/2}/C_D$, the same condition as minimum power. Gliding flight is the natural test case for the static stability of a flying wing, since a model must trim and recover from disturbances in a glide before any power is added.

13. Wing Planform Geometry

13.1 Definitions

The **planform** is the shape of the wing seen from above. The **span** b is the tip-to-tip distance. The **planform area** S is the projected area, conventionally including the part of the wing that passes through any fuselage. The **aspect ratio** is

$$AR = \frac{b^2}{S}$$

which for a rectangular wing is simply span divided by chord. The **root chord** c_r is the chord at the plane of symmetry, the **tip chord** c_t the chord at the tip, and the **taper ratio** $\lambda = c_t/c_r$. The **sweep angle** Λ is the angle between a spanwise line on the wing and the perpendicular to the plane of symmetry; it must always be stated with its reference line, usually the leading edge (Λ_{LE}), the quarter-chord line ($\Lambda_{c/4}$) or the half-chord line ($\Lambda_{c/2}$). The **dihedral angle** Γ_D is the upward inclination of the wing seen from the front. The **twist** $\varepsilon(y)$ is the incidence of a local section relative to the root; negative twist toward the tip, nose-down, is **washout**, and positive twist is **wash-in**.

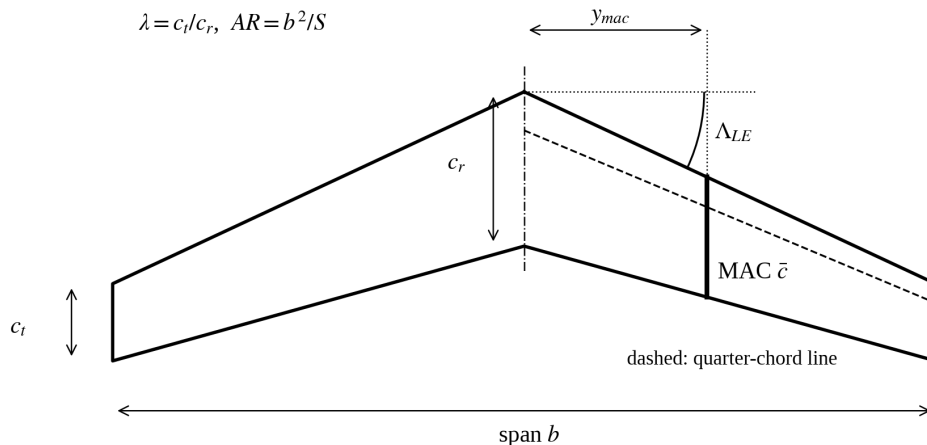


Figure 28. Geometry of a straight-tapered swept wing: span, root and tip chords, leading-edge sweep, the quarter-chord line, and the mean aerodynamic chord with its spanwise position.

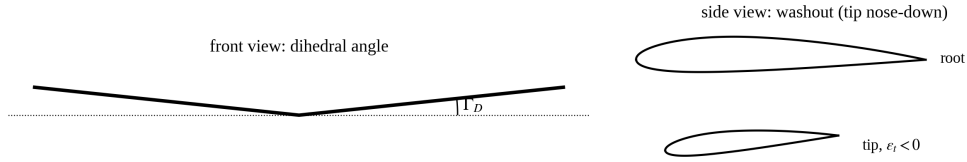


Figure 29. Left: dihedral seen from the front. Right: washout seen from the side, with the tip section set nose-down relative to the root.

13.2 The mean aerodynamic chord

The **mean aerodynamic chord (MAC)** $\bar{c}$ is the chord of an equivalent rectangular wing that would have the same lift and pitching moment characteristics. It is the reference length for C_m , and positions of the centre of gravity, aerodynamic centre and neutral point are quoted as fractions of it, measured from its leading edge. In general,

$$\bar{c} = \frac{2}{S} \int_0^{b/2} c(y)^2 dy$$

For a straight-tapered wing this gives closed-form results for the MAC, its spanwise position y_{mac} , and the position of its leading edge behind the root leading edge:

$$S = \frac{b}{2} c_r (1 + \lambda), \quad \bar{c} = \frac{2}{3} c_r \frac{1 + \lambda + \lambda^2}{1 + \lambda}, \quad y_{mac} = \frac{b}{6} \frac{1 + 2\lambda}{1 + \lambda}, \quad x_{LE, mac} = y_{mac} \tan \Lambda_{LE}$$

To a good approximation for wings of moderate aspect ratio and sweep, the aerodynamic centre of the whole wing lies at the quarter chord of the MAC. This is the practical starting point for locating the neutral point of a flying wing, and it explains why sweep moves the neutral point aft: the MAC itself moves aft as the wing is swept.

13.3 Converting between sweep angles

For a straight-tapered wing the sweep of any constant-percentage chord line n can be found from that of another line m :

$$\tan \Lambda_n = \tan \Lambda_m - \frac{4}{AR} \left[(n - m) \frac{1 - \lambda}{1 + \lambda} \right]$$

where n and m are chord fractions (0 for the leading edge, 0.25 for the quarter chord).

13.4 Worked example: the example flying wing

The example wing has $b = 1.6$ m, $c_r = 0.30$ m, $c_t = 0.15$ m and $\Lambda_{LE} = 25^\circ$. Then $\lambda = 0.5$ and $S = 0.8 \times 0.30 \times 1.5 = 0.36$ m², so $AR = 2.56/0.36 = 7.11$. The mean aerodynamic chord is $\bar{c} = (2/3)(0.30)(1.75/1.5) = 0.233$ m, located at $y_{mac} = (1.6/6)(2/1.5) = 0.356$ m from the centreline, with its leading edge $0.356 \tan 25^\circ = 0.166$ m behind the root leading edge. The quarter-chord sweep is $\tan^{-1}[\tan 25^\circ - (4/7.11)(0.25)(0.5/1.5)] = 22.8^\circ$. The estimated aerodynamic centre of the wing is at $0.166 + 0.25 \times 0.233 = 0.224$ m behind the apex.

Note the effect of the sweep. The tip is $0.8 \tan 25^\circ = 0.37$ m behind the root leading edge, so the tip region has a lever arm of roughly 0.15–0.2 m behind the aerodynamic centre. This is the geometric fact that allows washout of a swept tip to act like a tail.

14. Finite-Wing Aerodynamics: Vortices, Downwash and Induced Drag

A real wing has tips. This section explains the consequences, which are the trailing vortex system, downwash, induced drag and the reduction of the lift-curve slope, and develops Prandtl's lifting-line theory, the framework in which the elliptical and bell-shaped lift distributions are defined and compared.

14.1 The physical origin of the trailing vortices

A lifting wing has higher pressure below than above. At the tips nothing separates the two, so air flows around the tip from the lower surface to the upper. This spanwise flow is outward under the wing and inward over it, and where the two streams meet at the trailing edge they roll up into a pair of **trailing vortices**, one from each tip, rotating in opposite directions. Between the vortices they induce a downward velocity, the **downwash**, at the wing and behind it. Outboard of the tips they induce an upward velocity, the upwash, which formation-flying birds exploit.

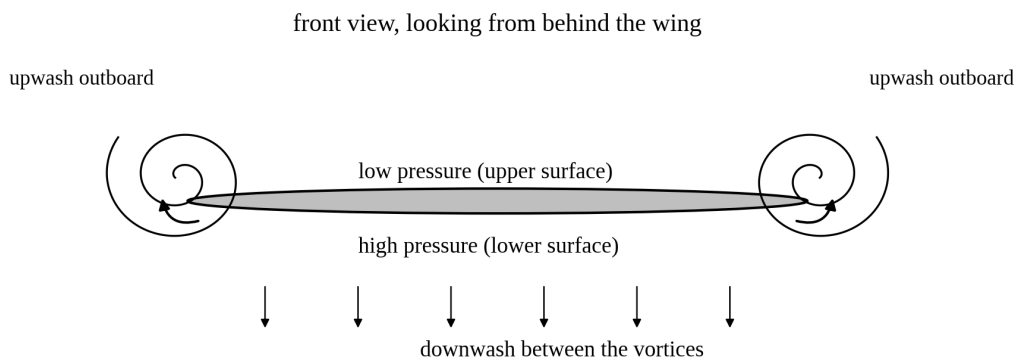


Figure 30. Air leaks around the wing tips from the high-pressure lower surface to the low-pressure upper surface and rolls up into two trailing vortices, which induce downwash between them.

14.2 Helmholtz's vortex theorems

Helmholtz's theorems govern vortex lines in inviscid flow. First, the strength of a vortex filament is constant along its length. Second, a vortex filament cannot end in the fluid: it must extend to infinity, end on a solid boundary, or form a closed loop. The second theorem is decisive for wings. The bound vortex that represents the wing's lift cannot simply stop at the tips, so it must turn and trail downstream. Wherever the bound circulation changes along the span, the change must be shed into the wake as trailing vorticity. This is why the wake of a real wing is a sheet of vorticity, which rolls up into the two concentrated tip vortices a short distance behind it.

14.3 The Biot–Savart law

The velocity induced at a point by a vortex filament is given by the **Biot–Savart law**, borrowed from electromagnetism. A segment $d\mathbf{l}$ of a filament of strength Γ induces, at a point displaced by $\mathbf{r}$ from it, the velocity

$$d\mathbf{V} = \frac{\Gamma}{4\pi} \frac{d\mathbf{l} \times \mathbf{r}}{|\mathbf{r}|^3}$$

For a straight segment from A to B and a point P at perpendicular distance h , with θ_1 and θ_2 the angles between the filament and the lines to P from each end, integration gives

$$V = \frac{\Gamma}{4\pi h} (\cos \theta_1 - \cos \theta_2)$$

An infinite straight vortex has $\theta_1 = 0$ and $\theta_2 = \pi$, giving $\Gamma/(2\pi h)$, the point-vortex result of Section 9.1. A **semi-infinite** vortex that starts at the foot of the perpendicular from P gives half of this, $\Gamma/(4\pi h)$. Every trailing vortex of a wing is a semi-infinite filament starting at the wing, and this result is the building block of lifting-line theory.

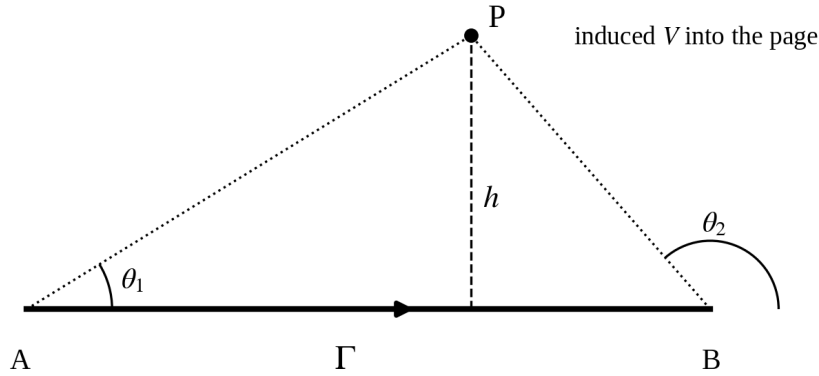


Figure 31. Geometry for the velocity induced by a straight vortex segment of strength Γ at a point P.

14.4 The horseshoe vortex

The simplest model of a wing is a single **horseshoe vortex**: a bound vortex of strength Γ along the span, joined at each tip to a trailing vortex running to infinity downstream. It obeys Helmholtz's theorems. The two trailing legs induce at a spanwise point y on the bound vortex a downwash

$$w(y) = \frac{\Gamma}{4\pi} \left(\frac{1}{b/2 + y} + \frac{1}{b/2 - y} \right)$$

which is finite at the centre but infinite at the tips. A single horseshoe is therefore unrealistic, but it shows the essential mechanism: trailing vortices induce downwash on the wing that produced them.

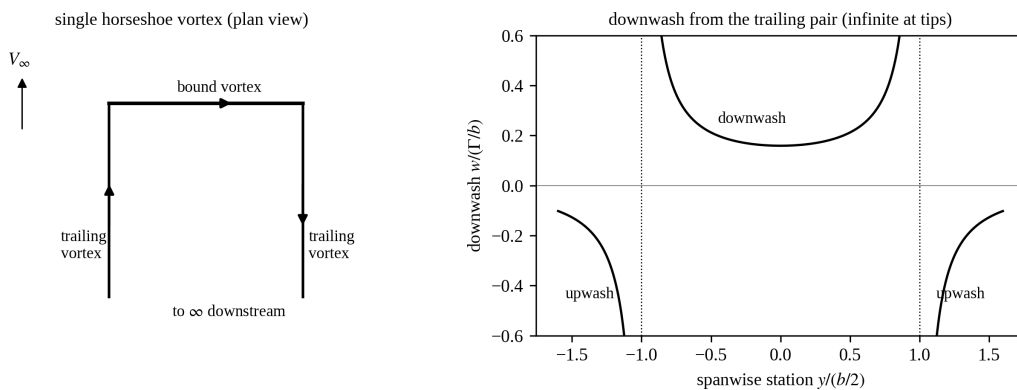


Figure 32. Left: a single horseshoe vortex. Right: the downwash it induces along the span, which becomes infinite at the tips and changes to upwash outboard of them.

14.5 Downwash, induced angle and induced drag

The downwash w tilts the flow that each section of the wing actually meets. The section sees a local relative wind inclined downward by the **induced angle of attack**

$$\alpha_i = \tan^{-1} \frac{w}{V_\infty} \approx \frac{w}{V_\infty}$$

so its **effective angle of attack** is $\alpha_{\text{eff}} = \alpha - \alpha_i$, smaller than the geometric angle. The section produces lift according to its own polar at α_{eff} and by Kutta–Joukowski this local lift is perpendicular to the local relative wind. It is therefore tilted rearward by α_i relative to the free stream. Resolving into free-stream axes gives a lift component and a drag component:

$$D'_i = L' \sin \alpha_i \approx L' \alpha_i$$

This is the **induced drag**. It is not caused by viscosity, and it is the price of producing lift with a wing of finite span. Physically, the energy it consumes appears as the kinetic energy of the swirling flow in the trailing vortex system. The same reasoning reversed is the basis of induced thrust in the flying-wing module: where a section sits in upwash, α_i is negative, the local lift is tilted forward, and the section produces a forward force.

14.6 Prandtl's lifting-line theory

Prandtl replaced the single horseshoe with an infinite number of horseshoes of infinitesimal strength, all with their bound segments on one spanwise line, the **lifting line**, normally placed at the quarter chord. The bound circulation $\Gamma(y)$ then varies smoothly along the span. By Helmholtz's theorem, wherever it changes, trailing vorticity of strength $-(d\Gamma/dy)dy$ is shed, forming a continuous vortex sheet. The downwash at a station y_0 is the sum of the contributions of all the semi-infinite trailing filaments:

$$w(y_0) = \frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy)dy}{y_0 - y}$$

At each station the local section obeys its two-dimensional lift curve at its effective angle of attack, and by Kutta–Joukowski its lift is $\rho_\infty V_\infty \Gamma$. Equating the two expressions for lift gives **Prandtl's lifting-line equation**:

$$\alpha(y_0) = \frac{2\Gamma(y_0)}{a_0 V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy)dy}{y_0 - y}$$

The three terms on the right are the angle needed by the section to carry its circulation, its zero-lift angle, and the induced angle. The local geometric angle $\alpha(y)$, which includes twist, the chord $c(y)$ and the section data a_0 and $\alpha_{L=0}(y)$ are given; the unknown is $\Gamma(y)$. The equation shows directly how planform (through c), geometric twist (through α) and aerodynamic twist (through $\alpha_{L=0}$) all shape the spanwise loading.

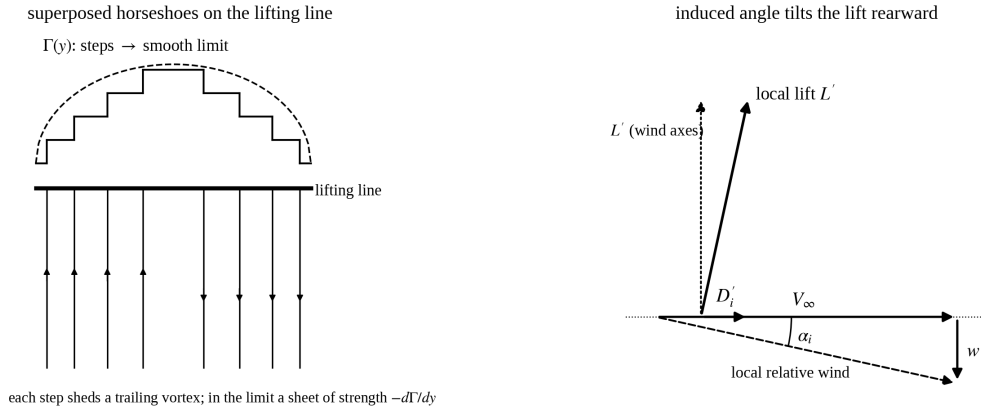


Figure 33. Left: the lifting line as a superposition of horseshoe vortices; in the limit each change in Γ sheds trailing vorticity. Right: the downwash tilts the local relative wind by the induced angle, which tilts the local lift rearward and creates induced drag.

14.7 The general solution: the monoplane equation

Glauert's substitution $y = -(b/2) \cos \theta$, with θ from 0 to π across the span, and a Fourier sine series for the circulation,

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^N A_n \sin n\theta$$

turn the lifting-line equation, with the Glauert integral, into a set of algebraic equations, the **monoplane equation**, to be satisfied at N chosen stations θ_0 :

$$\alpha(\theta_0) = \frac{4b}{a_0 c(\theta_0)} \sum_{n=1}^N A_n \sin n\theta_0 + \alpha_{L=0}(\theta_0) + \sum_{n=1}^N n A_n \frac{\sin n\theta_0}{\sin \theta_0}$$

The last term is the induced angle at each station. Once the coefficients are found, the wing's lift and induced drag follow from integrals that the orthogonality relations reduce to remarkably simple forms:

$$C_L = \pi A R A_1, \quad C_{D,i} = \pi A R \sum_{n=1}^N n A_n^2 = \frac{C_L^2}{\pi A R} (1 + \delta), \quad \delta = \sum_{n=2}^N n \left(\frac{A_n}{A_1} \right)^2$$

Only the first coefficient contributes to the lift, but every coefficient adds to the drag, since $\delta \geq 0$. The **span efficiency factor** is $e = 1/(1 + \delta)$. This is one of the most important results in aerodynamics: for a given span and lift, the induced drag is least when all the coefficients beyond the first vanish, that is, when the circulation is purely $\sin \theta$, which is elliptical in y .

14.8 The elliptical lift distribution

With only A_1 non-zero, the circulation is

$$\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b} \right)^2}, \quad L = \frac{\pi}{4} \rho_\infty V_\infty b \Gamma_0$$

and the induced angle is the same at every station: the downwash is **uniform** along the span, $w = \Gamma_0/(2b)$, and

$$\alpha_i = \frac{C_L}{\pi AR}, \quad C_{D,i} = \frac{C_L^2}{\pi AR}, \quad e = 1$$

An untwisted wing with an elliptical planform produces this loading, and so does any planform with a suitable twist distribution, at one particular lift coefficient. Uniform downwash also means a uniform local lift coefficient along the span, so an untwisted elliptical wing stalls everywhere at once. The induced drag, inversely proportional to aspect ratio, falls as span increases, which is why sailplanes have long wings.

The result states the optimum *for a given span*. Prandtl's 1933 bell-shaped distribution, discussed in the flying-wing module, answers a different question: what is the least induced drag for a given lift and a given structural requirement, with the span left free? Section 17 explains the structural side of that comparison.

14.9 Non-elliptical wings: taper and the local lift coefficient

Solving the monoplane equation for straight-tapered untwisted wings of aspect ratio 7 gives span efficiencies of $e = 0.945$ for a rectangular wing, 0.989 for $\lambda = 0.4$ and 0.875 for a pointed tip ($\lambda = 0$). A moderate taper ratio of about 0.35–0.45 gives a loading very close to elliptical on an unswept wing, which is the aerodynamic justification for taper discussed in the flying-wing module. The local lift coefficient, however, tells a different story. A rectangular wing has its highest local c_l at the root and stalls there first, which is benign. A strongly tapered wing has its highest local c_l toward the tip, and with the lower tip Reynolds number this makes tip stall likely. Aft sweep shifts the loading further outboard and worsens the tendency (Section 15.2).

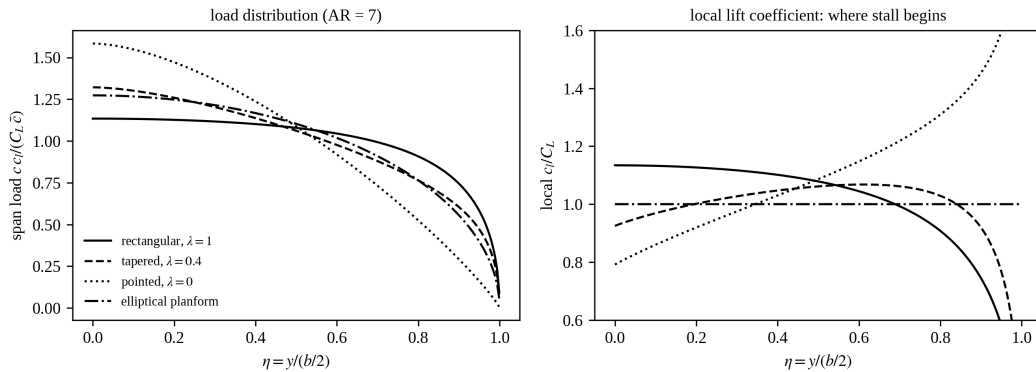


Figure 34. Lifting-line solutions for untwisted wings of aspect ratio 7 at the same angle of attack. Left: span loading, normalised by the wing lift coefficient and mean chord. Right: local section lift coefficient, which shows that strong taper raises the tip loading and so promotes tip stall.

A quick approximation to the span loading of a wing without sweep, useful for preliminary structural sizing, is **Schrenk's approximation**, which takes the load to be the average of the planform shape and an ellipse of the same area:

$$c c_l(y) \approx \frac{C_L}{2} \left[c(y) + \frac{4S}{\pi b} \sqrt{1 - \left(\frac{2y}{b} \right)^2} \right]$$

14.10 Twist: basic and additional loading

Because the lifting-line equation is linear, the span loading of a twisted wing can be split into two parts. The **additional loading** is the loading of the same wing without twist, and it is proportional to C_L . The **basic loading** is the loading caused by the twist alone at the angle of attack for which the total lift is zero; it does not change with C_L . With washout, the basic loading is positive inboard and negative outboard. The total loading is the sum. This decomposition explains several statements in the flying-wing module at once. Washout unloads the tips at every lift coefficient, but its relative effect is largest at low C_L , where the tips may carry a download. Twist can make the loading of any planform elliptical, or bell-shaped, but only at one design lift coefficient. And on a swept wing the negative basic loading at the tips, acting far behind the aerodynamic centre, produces a nose-up moment even when the total lift is zero, which is the positive $C_{m,ac}$ that trims a flying wing.

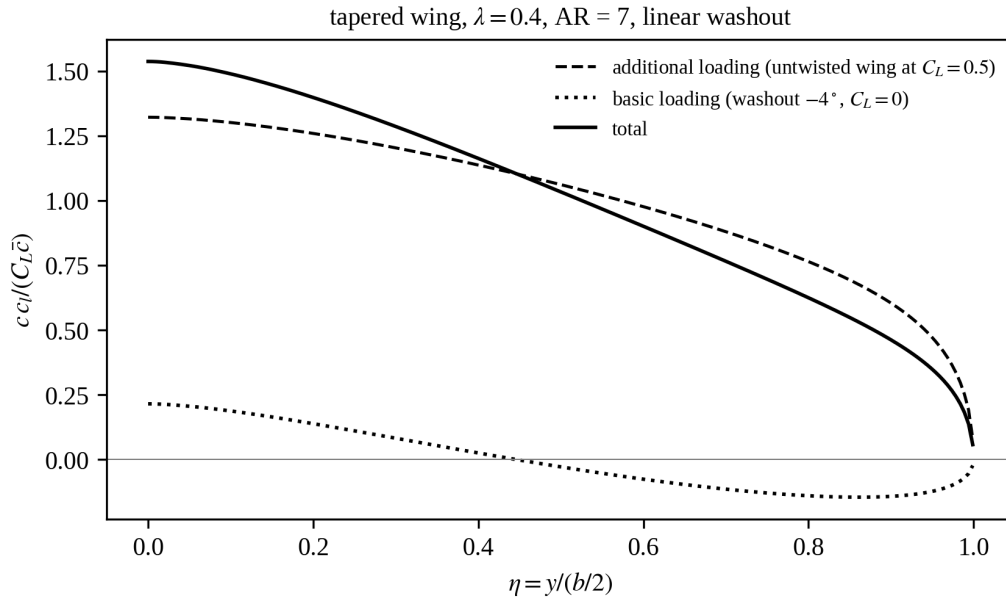


Figure 35. Basic and additional loadings of a tapered wing with 4° of linear washout, computed by lifting-line theory. The basic loading integrates to zero lift but is negative at the tips; added to the additional loading it unloads the outer wing.

14.11 The lift-curve slope of a finite wing

Because the induced angle grows with lift, a finite wing needs a larger geometric angle of attack than an airfoil to reach the same lift coefficient, and its lift-curve slope is lower. For an elliptical wing, $\alpha = \alpha_{\text{eff}} + C_L/(\pi AR)$, which gives

$$C_{L,\alpha} = \frac{a_0}{1 + a_0/(\pi AR)}$$

and for other planforms a small correction factor τ appears, $C_{L,\alpha} = a_0/[1 + (a_0/\pi AR)(1 + \tau)]$.

Lifting-line theory becomes inaccurate for aspect ratios below about 4 and for swept wings. A widely used semi-empirical formula that covers both, known as the Helmbold or DATCOM formula, is

$$C_{L,\alpha} = \frac{2\pi AR}{2 + \sqrt{4 + AR^2(1 + \tan^2 \Lambda_{c/2})}}$$

for incompressible flow with $a_0 = 2\pi$. For the example wing, with $AR = 7.11$ and $\Lambda_{c/2} = 20.4^\circ$, this gives 4.54 per radian (0.079 per degree), about 72 percent of the section value. Lifting-line theory

gives 4.84 per radian for the same planform without sweep, with a span efficiency of 0.985.

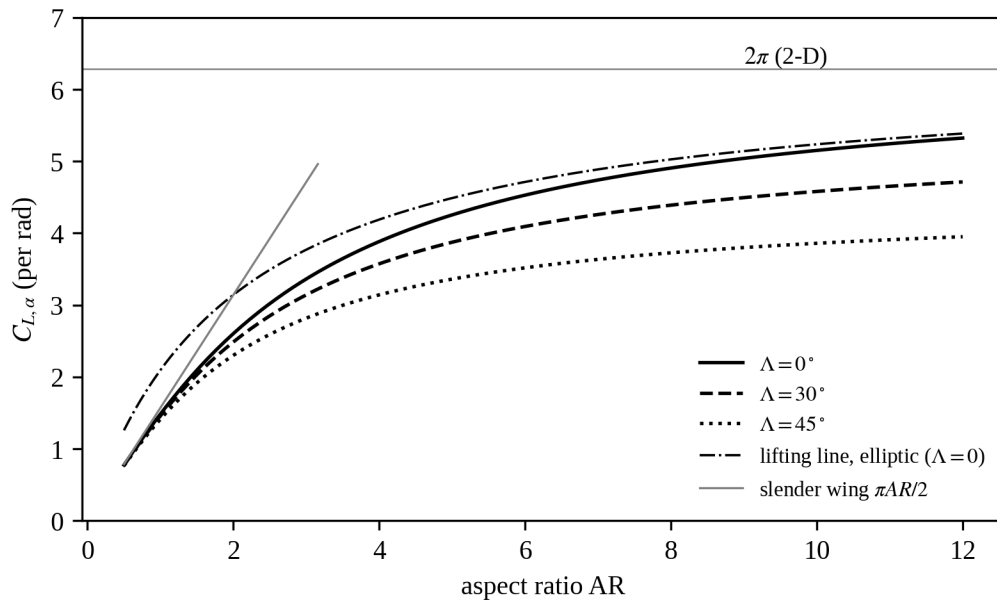


Figure 36. Lift-curve slope of a finite wing against aspect ratio, from the Helmbold–DATCOM formula for three half-chord sweep angles, compared with elliptical lifting-line theory and slender-wing theory. Low aspect ratio and sweep both reduce the slope.

14.12 Numerical methods: the vortex-lattice method

Lifting-line theory treats each section as two-dimensional and places all the bound vorticity on one line, so it cannot represent sweep or low aspect ratio well. The **vortex-lattice method** (VLM) removes these limits. The wing's mean surface is divided into a grid of panels. On each panel a horseshoe vortex is placed with its bound segment at the panel's quarter-chord line and its trailing legs running downstream. A **collocation point** is placed at the panel's three-quarter-chord point, where the condition of zero flow through the surface is enforced. (The quarter-chord–three-quarter-chord arrangement reproduces exactly the thin-airfoil result for a flat plate and builds in the Kutta condition.) Using the Biot–Savart law, the velocity induced at every collocation point by every horseshoe is computed, which gives a set of linear equations for the unknown strengths. The forces follow from Kutta–Joukowski on each bound segment. Programs such as AVL and XFLR5 implement this method; the "linear VLM analysis" mentioned in the flying-wing module is exactly this. VLM gives span loading, induced drag, the neutral point and the stability derivatives, but because it is inviscid it cannot predict profile drag or stall, which must be added from the section polars.

vortex-lattice discretisation of a half wing

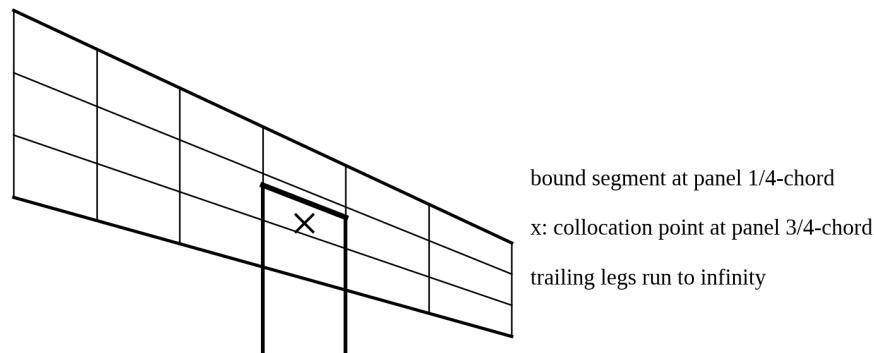


Figure 37. Vortex-lattice discretisation of a swept, tapered half wing. Each panel carries a horseshoe vortex whose bound segment lies on the panel quarter-chord line; the flow tangency condition is imposed at the three-quarter-chord collocation point.

14.13 What the flying-wing module uses from this section

Downwash, induced angle and induced drag explain why lift distribution matters. The monoplane equation and its drag result define the elliptical optimum and the span efficiency. Basic and additional loading explain how washout trims and stabilises a swept wing and why its effect depends on lift coefficient. The local lift-coefficient distribution explains tip stall on tapered wings. Upwash and the forward tilt of local lift in upwash are the mechanism of induced thrust and proverse yaw. The vortex-lattice method is the analysis tool.

15. Sweep

15.1 Simple sweep theory

Consider an infinitely long wing swept at angle Λ . The free stream can be split into a component normal to the leading edge, $V_\infty \cos \Lambda$, and a component parallel to it, $V_\infty \sin \Lambda$. On an infinite wing the parallel component meets no change of shape and so, in inviscid flow, has no effect on the pressures. The section pressures are set by the normal component alone. This simple model, which is accurate for the mid-span of a long swept wing, has three consequences. The effective dynamic pressure is reduced by $\cos^2 \Lambda$, which lowers the lift-curve slope (approximately by $\cos \Lambda$ in terms of the free-stream angle of attack). The effective Mach number is reduced to $M \cos \Lambda$, which raises the critical Mach number; this is the reason for sweep on high-speed aircraft, mentioned in the flying-wing module. And the spanwise component of the flow, although it does not affect the inviscid pressures, does affect the boundary layer.

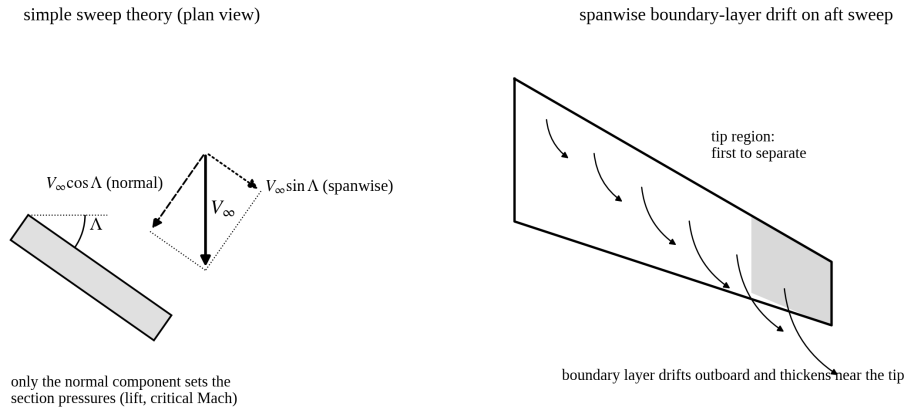


Figure 38. Left: simple sweep theory; only the velocity component normal to the leading edge sets the section pressures. Right: on an aft-swept wing the slow boundary-layer fluid drifts outboard and accumulates near the tips, which promotes tip stall.

15.2 Spanwise flow and tip loading on swept wings

The boundary layer has less momentum than the outer flow, so it is more easily turned by spanwise pressure gradients. On an aft-swept wing the isobars are themselves swept, and the slow fluid near the surface drifts outboard along them. Boundary-layer fluid accumulates toward the tips, thickening the tip boundary layer and making it more prone to separation. In addition, each part of an aft-swept wing lies in the upwash field of the parts ahead of and inboard of it, which increases the loading of the outer wing. Both effects make aft-swept wings prone to **tip stall**, which is the background to the statement in the flying-wing module that swept wings tend to stall at the tips first because of lateral boundary-layer flow. Washout, which lowers the tip angle of attack, is the principal remedy. Physical fences on the upper surface are another, used on many swept-wing aircraft to block the spanwise drift.

15.3 Sweep and the pitching moment

Sweep also changes the longitudinal geometry. On an aft-swept wing the outer sections lie behind the inner ones, so the aerodynamic centre of the whole wing, at the quarter chord of the MAC, lies behind the root quarter chord, and the outer wing acts through a long lever arm about it. Any change of load on the outer wing therefore produces a large pitching moment. If the tips lose lift, as in a tip stall, the remaining lift is concentrated on the inner, forward part of the wing, and the aircraft pitches nose-up, which deepens the stall. If the tips carry less lift than the inner wing in normal flight because of washout, the nose-up moment that results is exactly what a tailless aircraft needs for trim. Sweep is therefore what gives washout its longitudinal leverage, and the flying-wing module's twist-sizing coefficient K grows with sweep for this reason.

16. Flight-Mechanics Foundations

16.1 Axis systems and angles

Flight mechanics uses several right-handed axis systems. The **Earth axes** are fixed to the ground, with z pointing down. The **body axes** are fixed to the aircraft with their origin at the centre of gravity: x forward along a reference line, y toward the right wing tip, and z downward. The **wind** (or stability) **axes** have x aligned with the relative wind. The orientation of the body relative to the Earth is given by three Euler angles: the heading or yaw angle ψ , the pitch attitude θ and the bank angle ϕ . The orientation of the relative wind relative to the body is given by two aerodynamic angles: the angle of attack α in the vertical plane of symmetry, and the sideslip angle β out of it. In wings-level flight without sideslip, the pitch attitude is the sum of the flight-path angle γ and the angle of attack, $\theta = \gamma + \alpha$.

The forces along the body axes are written X, Y, Z , and the moments about them L, M, N : the rolling, pitching and yawing moments, positive right wing down, nose up and nose right respectively. The angular rates are p, q, r . The rolling moment shares its symbol with lift, and many texts write it as ℓ to avoid confusion; context usually makes the meaning clear. The corresponding coefficients are $C_l = L/(qSb)$, $C_m = M/(qSc)$ and $C_n = N/(qSb)$, with the span as reference length for the lateral moments.

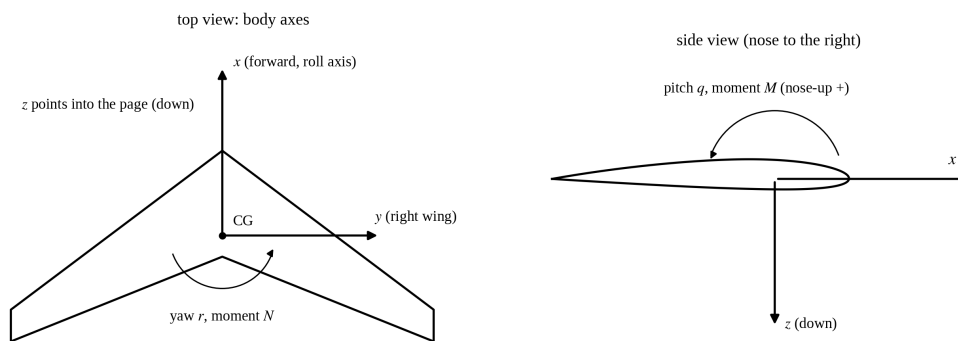
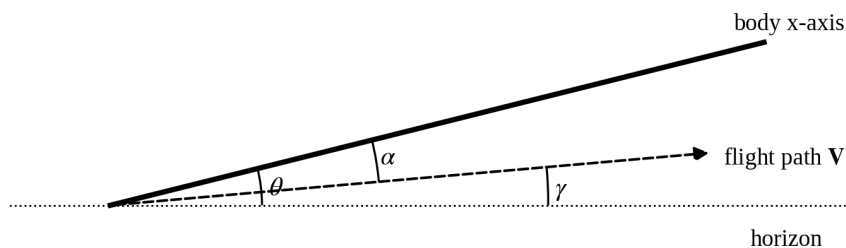


Figure 39. Body axes with origin at the centre of gravity. The x axis points forward, y toward the right wing and z downward. Rolling, pitching and yawing moments L, M, N act about x, y and z .



$$\theta = \gamma + \alpha \quad (\text{wings level, no sideslip})$$

Figure 40. Angles in the plane of symmetry: pitch attitude θ of the body axis, flight-path angle γ of the velocity, and angle of attack α between them.

16.2 Control surfaces

A conventional aircraft controls pitch with an **elevator** on the horizontal tail, roll with **ailerons** near the wing tips, deflected in opposite directions, and yaw with a **rudder** on the fin. A trailing-edge surface deflected downward increases the camber and lift of the sections ahead of it; deflected upward it decreases them. A flying wing has no tail, so the surfaces at the trailing edge of the outer wing do both jobs and are called **elevons**. Deflected together (symmetrically) they act as an elevator: because they are behind the aerodynamic centre on a swept wing, raising both produces a nose-up moment. Deflected differentially (antisymmetrically) they act as ailerons. With δ_L and δ_R the left and right deflections, positive trailing-edge down, the equivalent elevator and aileron inputs are

$$\delta_e = \frac{\delta_L + \delta_R}{2}, \quad \delta_a = \frac{\delta_L - \delta_R}{2}$$

and a mixer in the flight controller combines the pilot's pitch and roll commands in this way. Without a rudder, yaw control on aircraft such as the B-2 is provided by **split drag rudders** (decelerons), surfaces at the tips that split open above and below to create drag on one side.

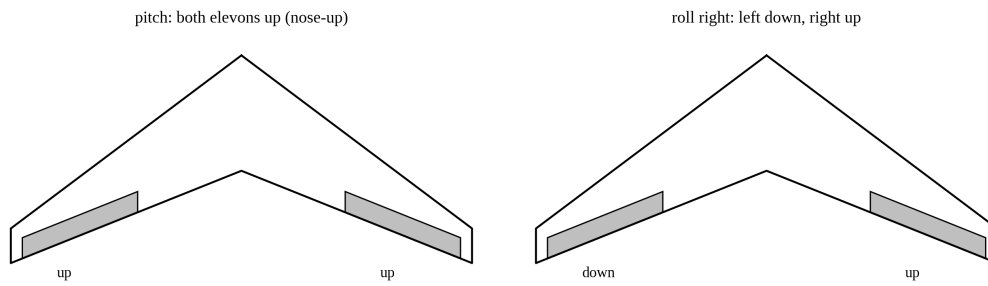


Figure 41. Elevons on a flying wing. Symmetric deflection controls pitch; differential deflection controls roll. For a roll to the right the left elevon goes down and the right elevon goes up.

16.3 Steady flight, wing loading and stall speed

In steady level flight, lift balances weight and thrust balances drag: $L = W$ and $T = D$. Since $L = \frac{1}{2}\rho V^2 SC_L$, the speed needed at a given lift coefficient is

$$V = \sqrt{\frac{2W}{\rho SC_L}}, \quad V_s = \sqrt{\frac{2W}{\rho SC_{L,max}}}$$

where V_s is the **stall speed**. The **wing loading** W/S is therefore the single most important number for an aircraft's speed range. A flying wing usually has a lower maximum lift coefficient than a tailed aircraft, because it cannot use powerful trailing-edge flaps (deflecting them would produce a large nose-down moment with no tail to balance it) and because its reflex or washout costs lift. For the example aircraft ($W/S = 54.5 \text{ N/m}^2$) with a modest $C_{L,max} = 0.9$, the stall speed at sea level is $\sqrt{(2 \times 54.5 / (1.225 \times 0.9))} = 9.9 \text{ m/s}$, and cruise at $C_L = 0.3$ is at 17.2 m/s .

16.4 Turning flight

In a level, coordinated turn at bank angle ϕ , the vertical component of lift supports the weight and the horizontal component provides the centripetal force. The **load factor** $n = L/W$ and the turn radius are

$$n = \frac{1}{\cos \phi}, \quad R_{turn} = \frac{V^2}{g \tan \phi}$$

and the stall speed in the turn rises to $V_s \sqrt{n}$. At 60° of bank $n = 2$ and the stall speed rises by 41 percent. A turn is also where yaw coupling matters: to roll into the turn the aircraft must deflect its ailerons, and any yawing moment that the ailerons create either helps or hinders the turn.

16.5 Adverse yaw

When the ailerons are deflected to roll right, the left aileron goes down and the right aileron goes up. The left wing produces more lift and the right wing less. On a wing with the usual downwash distribution, the extra lift on the left wing is tilted rearward by the local induced angle, so the left wing also produces more induced drag, and the down-going aileron adds profile drag as well. The extra drag on the rising left wing yaws the nose to the left, opposite to the intended turn. This is **adverse yaw**. A conventional aircraft corrects it with rudder; a fin also helps by weathercocking. A flying wing without a fin has neither, which is why the flying-wing module devotes attention to spanwise loadings, such as the bell shape, whose tips sit in upwash and therefore produce the opposite, favourable effect, **proverse yaw**.

16.6 The centre of gravity

The **centre of gravity** (CG) is the point at which the weight can be considered to act. For an aircraft assembled from components of mass m_i at positions x_i ,

$$x_{cg} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

It is expressed as a fraction of the MAC, $h_{cg} = (x_{cg} - x_{LE,mac})/\bar{c}$. Its position is chosen by the designer, mainly by placing the battery, and it is the single most important parameter for the stability of a flying wing.

Worked example. The components of the example aircraft, with positions measured aft of the root leading edge, are listed below.

Component	Mass (kg)	Position x (m)	m·x (kg·m)
Wing structure and skins	0.90	0.26	0.2340
Battery	0.45	0.13	0.0585
Motor and propeller (pusher)	0.15	0.30	0.0450
Receiver, controller, wiring	0.10	0.15	0.0150
Servos	0.06	0.28	0.0168
Payload	0.34	0.12	0.0408
Total	2.00		0.4101

Table 2. Mass breakdown of the example flying wing.

The CG is at $x_{cg} = 0.4101/2.00 = 0.205$ m behind the apex. With the MAC leading edge at 0.166 m and $\bar{c} = 0.233$ m (Section 13.4), $h_{cg} = (0.205 - 0.166)/0.233 = 0.17$. With the neutral point estimated at the quarter chord of the MAC, the static margin is $0.25 - 0.17 = 0.08$, or 8 percent, a typical value for a

flying wing. Moving the 0.45 kg battery 20 mm aft moves the CG by $0.45 \times 0.02/2.0 = 4.5$ mm, or 2 percent of the MAC, which shows how sensitive the static margin is to battery placement.

16.7 Equilibrium, trim and the concept of stability

An aircraft is in **equilibrium**, or **trimmed**, when the net force and the net moment about its centre of gravity are zero. Trim is about balance; stability is about what happens after a disturbance. **Static stability** concerns the initial tendency: if, after a small disturbance, the forces and moments that arise tend to return the aircraft toward equilibrium, it is statically stable; if they tend to increase the disturbance, it is statically unstable; if they are zero, it is neutrally stable. The ball in a bowl, on a hill and on a flat surface is the classic illustration.

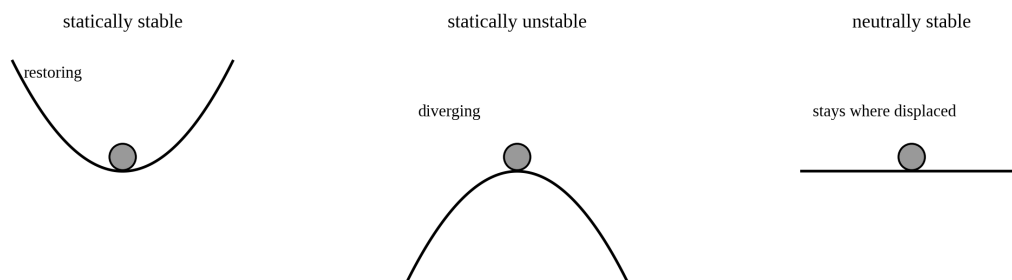


Figure 42. Static stability. A displaced ball in a bowl experiences a restoring force; on a hill it experiences a diverging force; on a flat surface it experiences none.

Dynamic stability concerns the whole motion over time. A statically stable aircraft is pushed back toward equilibrium, but it may overshoot and oscillate. If the oscillations die away it is dynamically stable; if they persist unchanged it is dynamically neutral; if they grow it is dynamically unstable. Static stability is necessary for dynamic stability in the classical modes but not sufficient. The flying-wing module deals only with static stability and defers the dynamic modes, which are the short-period and phugoid modes in pitch and the roll subsidence, spiral and Dutch roll modes laterally, to its follow-on module.

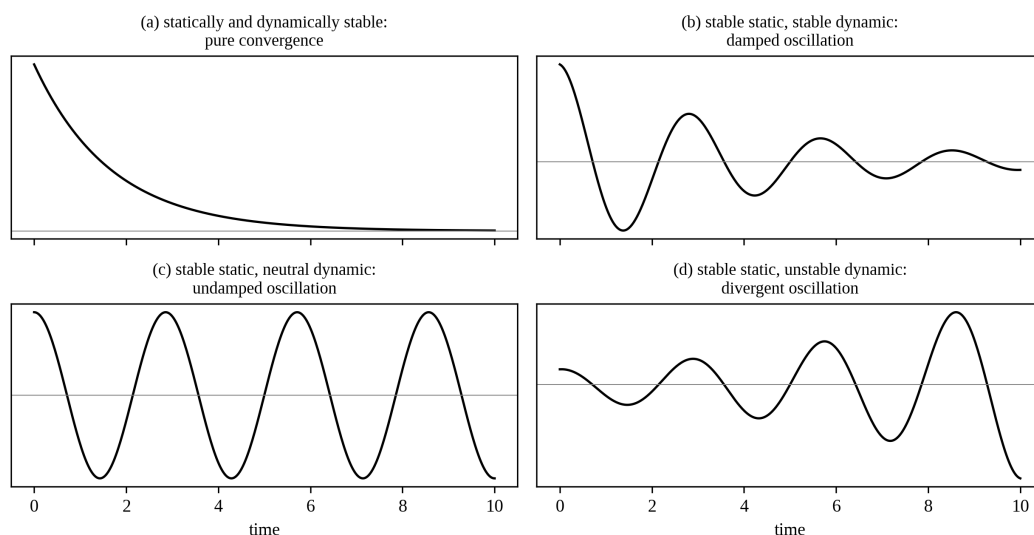


Figure 43. Possible responses after a disturbance of a statically stable system: (a) pure convergence, (b) damped oscillation, (c) undamped oscillation, (d) divergent oscillation.

16.8 Stability derivatives

Linearising the aerodynamic moments about a trimmed condition (Section 1.3) expresses each moment change as a sum of partial derivatives times the disturbances. These derivatives are called **stability derivatives**, and their signs decide the static stability in each axis. The three most important static derivatives are the following. In pitch, $C_{m,\alpha} = \partial C_m / \partial \alpha$ must be negative, so that a nose-up disturbance produces a nose-down moment. In yaw, the **weathercock** or directional stability derivative $C_{n,\beta} = \partial C_n / \partial \beta$ must be positive, so that a sideslip makes the nose turn into the relative wind. In roll, the **dihedral effect** $C_{l,\beta} = \partial C_l / \partial \beta$ must be negative, so that a sideslip produces a rolling moment that raises the leading wing and levels the aircraft. Geometric dihedral and aft sweep both make $C_{l,\beta}$ more negative, which is why swept flying wings often need little or no dihedral. A fin provides most of $C_{n,\beta}$ on a conventional aircraft; without one, directional stability must come from sweep, winglets, the spanwise loading or active control. Control derivatives, such as $\partial C_n / \partial \delta_a$, measure the side effects of controls; its sign distinguishes adverse from proverse yaw.

16.9 Pitch stability of a conventional aircraft and the neutral point

To appreciate what a flying wing must do without a tail, it helps to see what a tail does. Take moments about the CG for a wing-body combination with a horizontal tail of area S_t whose aerodynamic centre is a distance l_t behind the CG. Using the transfer formula of Section 8.3 for the wing and the lever arm for the tail,

$$C_{m,cg} = C_{m,ac,wb} + C_{L,wb}(h_{cg} - h_{ac,wb}) - V_H \eta_t C_{L,t}, \quad V_H = \frac{l_t S_t}{S \bar{c}}$$

where V_H is the **tail volume coefficient** and η_t the ratio of dynamic pressure at the tail to the free stream. The tail sits in the downwash of the wing, at angle ϵ , so its angle of attack is $\alpha - \epsilon$ plus a fixed setting, and ϵ grows with α at the rate $d\epsilon/d\alpha$. With a and a_t the lift-curve slopes of the wing-body and the tail, differentiating with respect to α gives

$$C_{m,\alpha} = a(h_{cg} - h_{ac,wb}) - V_H \eta_t a_t \left(1 - \frac{d\epsilon}{d\alpha}\right)$$

The **neutral point** is the CG position at which this vanishes:

$$h_{np} = h_{ac,wb} + V_H \eta_t \frac{a_t}{a} \left(1 - \frac{d\epsilon}{d\alpha}\right)$$

and the stability condition becomes $h_{cg} < h_{np}$, with the **static margin** $h_{np} - h_{cg}$ measuring the degree of stability. The tail moves the neutral point aft of the wing's aerodynamic centre, which lets the CG sit behind the wing's AC while the aircraft remains stable, and a tail download balances the wing's negative $C_{m,ac}$ for trim. A tailless aircraft has $V_H = 0$, so its neutral point is simply the aerodynamic centre of the wing, and its trim moment must come from the wing itself. That is exactly the starting point of Section 2.4 of the flying-wing module.

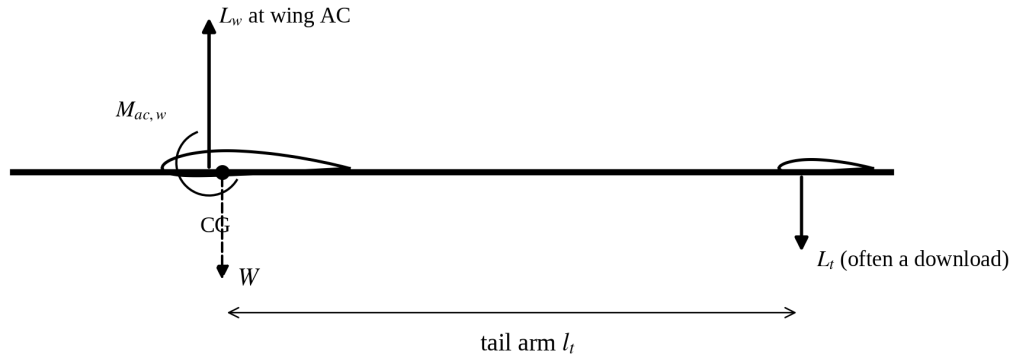


Figure 44. Forces and moments on a conventional aircraft. The tail, on a long arm behind the CG, provides both the trimming moment and the stabilising contribution that a flying wing must generate within its own wing.

17. Structural Foundations for Wing Design

The bell-shaped lift distribution of the flying-wing module is optimal only when structural weight is taken into account, so a minimal understanding of wing bending is required.

17.1 The wing as a cantilever beam

Each half of a wing behaves as a cantilever beam built in at the root and loaded by the net upward load per unit span $q(y)$, the lift minus the weight of the structure and anything carried in the wing. Cutting the wing at a station y and considering everything outboard of the cut, the **shear force** and **bending moment** carried across the cut are

$$V(y) = \int_y^{b/2} q(s) ds, \quad M(y) = \int_y^{b/2} q(s)(s - y) ds$$

or, in differential form, $dV/dy = -q$ and $dM/dy = -V$. Both are greatest at the root. The root bending moment of one semispan is the moment of the semispan's load about the centreline:

$$M_{root} = \int_0^{b/2} q(y) y dy$$

17.2 Bending moment for different span loadings

For the same total lift L and the same span, the root bending moment depends on how far outboard the lift acts. Writing the result in terms of the spanwise centroid of the semispan load gives, for three loadings,

Span loading	Centroid of semispan load	Root bending moment
Uniform	0.500 ($b/2$)	0.125 Lb
Elliptical, $(1 - \eta^2)^{1/2}$	$4/(3\pi) = 0.424$ ($b/2$)	$Lb/(3\pi) = 0.106$ Lb
Bell-shaped, $(1 - \eta^2)^{3/2}$	$16/(15\pi) = 0.340$ ($b/2$)	0.085 Lb

Table 3. Root bending moment of one semispan, for equal total lift and equal span.

The bell-shaped loading has 20 percent less root bending moment than the elliptical loading at the same span. Conversely, at the same root bending moment it can have a longer span, and a longer span reduces induced drag in proportion to $1/b^2$. This is the trade that makes the bell shape superior when

structure is the constraint. How the structural constraint is defined changes the exact numbers. Prandtl's 1933 analysis fixed the integral of the bending moment along the span, $\int M dy$, which is a better proxy for spar weight than the root moment alone. Integrating by parts twice shows that this integral equals $\frac{1}{2} \int q(y) y^2 dy$, the second moment of the load distribution. Other treatments fix the root bending moment. This is why quoted figures for the span increase and drag reduction of the bell-shaped loading differ between sources, as the flying-wing module notes.

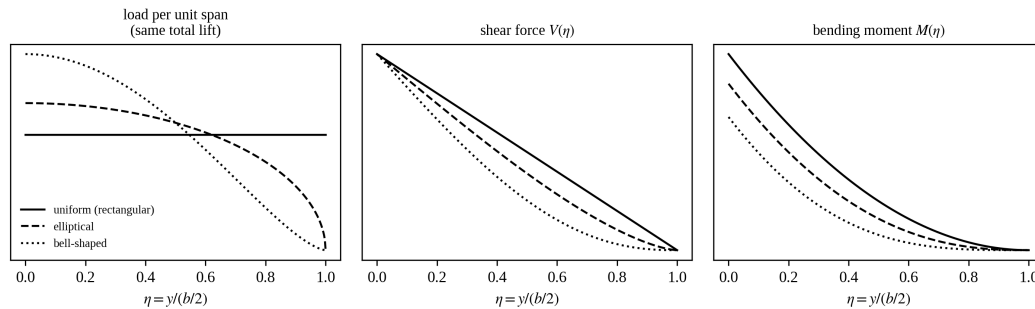


Figure 45. Load, shear force and bending moment along the semispan for uniform, elliptical and bell-shaped loadings of equal total lift. Moving load inboard reduces the shear and bending moment everywhere.

17.3 Bending stress and structural weight

The bending moment is resisted by the spar. In elementary beam theory the bending stress at distance z from the neutral axis of a section with second moment of area I is

$$\sigma = \frac{Mz}{I}$$

For a spar with two caps separated by depth h , the caps carry forces $\pm M/h$, so the cap area needed at an allowable stress σ_{allow} is $A = M/(h\sigma_{\text{allow}})$, and the cap mass is proportional to $\int M/h dy$. Structural weight therefore grows with the bending moment and falls with the depth of the section. Thick root sections, inboard loading and a short span all reduce spar weight. A flying wing also benefits from **span loading**: carrying batteries, fuel and payload spread along the span places weight where the lift is, which reduces the net load $q = \text{lift} - \text{weight}$ and so the bending moment.

17.4 Torsion and aeroelastic twist

The pitching moment about the aerodynamic centre, together with any offset between the line of aerodynamic centres and the wing's shear centre, twists the wing. On an aft-swept wing there is also a geometric coupling between bending and twist: when the wing bends upward under load, the streamwise sections near the tip rotate nose-down, so the wing washes out further as the load increases. This **aeroelastic washout** unloads the tips at high load factor, which is stabilising in pitch but reduces tip lift and roll authority. A very flexible wing can even suffer control reversal, in which the twisting caused by an aileron deflection cancels or reverses its intended effect. For a model-scale flying wing these effects are usually small if the wing is torsionally stiff, but they are the reason designers specify a stiff torsion box, for example a closed D-tube leading edge, rather than relying on a single spar.

18. Summary of Key Relations

The principal results of this module are collected here for reference, grouped by the section in which they are derived.

Fluid properties and flow

$$p = \rho RT, \quad a = \sqrt{\gamma RT}, \quad Re = \frac{\rho V l}{\mu}, \quad M = \frac{V}{a}$$

$$\nabla \cdot \mathbf{V} = 0, \quad \Gamma = -\oint_C \mathbf{V} \cdot d\mathbf{s}, \quad p + \frac{1}{2}\rho V^2 = p_0, \quad C_p = 1 - \left(\frac{V}{V_\infty}\right)^2$$

Forces, moments and reference points

$$L' = N' \cos \alpha - A' \sin \alpha, \quad D' = N' \sin \alpha + A' \cos \alpha$$

$$c_{m,2} = c_{m,1} + (h_2 - h_1)c_l, \quad h_{cp} = h_{ac} - \frac{c_{m,ac}}{c_l}, \quad h_{ac} = -\frac{dc_{m,LE}/d\alpha}{dc_l/d\alpha}$$

Lift theory

$$L' = \rho_\infty V_\infty \Gamma, \quad c_l = 2\pi(\alpha - \alpha_{L=0}), \quad c_{m,c/4} = \frac{\pi}{4}(A_2 - A_1)$$

$$\alpha_{L=0} = -\frac{1}{\pi} \int_0^\pi \frac{dz}{dx} (\cos \theta_0 - 1) d\theta_0$$

Finite wings

$$AR = \frac{b^2}{S}, \quad \bar{c} = \frac{2}{3} c_r \frac{1 + \lambda + \lambda^2}{1 + \lambda}, \quad \alpha_i = \frac{w}{V_\infty}, \quad D'_i = L' \alpha_i$$

$$C_L = \pi AR A_1, \quad C_{D,i} = \frac{C_L^2}{\pi AR e}, \quad e = \frac{1}{1 + \delta}, \quad C_{L,\alpha} = \frac{a_0}{1 + a_0/(\pi AR)}$$

Performance

$$C_D = C_{D0} + KC_L^2, \quad \left(\frac{L}{D}\right)_{max} = \frac{1}{2\sqrt{KC_{D0}}}, \quad V_s = \sqrt{\frac{2W}{\rho S C_{L,max}}}, \quad \tan \gamma = \frac{1}{L/D}$$

Stability and structure

$$x_{cg} = \frac{\sum m_i x_i}{\sum m_i}, \quad C_{m,\alpha} < 0, \quad h_{np} = h_{ac,wb} + V_H \eta_t \frac{a_t}{a} \left(1 - \frac{d\epsilon}{d\alpha}\right), \quad SM = h_{np} - h_{cg}$$

$$M_{root} = \int_0^{b/2} q(y) y dy, \quad \sigma = \frac{Mz}{I}$$

19. Review Questions

1. Show that the moment of a force about a point depends only on the component of the force perpendicular to the position vector. Why is a pitching moment conventionally positive nose-up?
2. Write the first-order Taylor expansion of $C_m(\alpha, \delta_e)$ about a trimmed condition and identify the stability and control derivatives in it.

3. Use Stokes's theorem to explain why the circulation around any contour enclosing a lifting airfoil is the same, while that around a contour not enclosing it is zero.
4. Evaluate the Fourier coefficients of $\Gamma(\theta) = \sin^3 \theta$ and show that the expansion contains only $n = 1$ and $n = 3$.
5. Find the air density and the speed of sound at 3000 m in the ISA. (Answer: 0.909 kg/m^3 ; 328.6 m/s .)
6. State the four conditions under which Bernoulli's equation applies between any two points of a flow. Why does it fail across a propeller disc?
7. A pitot-static probe at sea level reads a pressure difference of 180 Pa. What is the airspeed? (Answer: 17.1 m/s .)
8. Compute the chord Reynolds numbers at the root and tip of the example flying wing at 17 m/s and comment on the consequences for the tip airfoil's polar.
9. Explain why a turbulent boundary layer has higher skin friction than a laminar one but resists separation better.
10. Describe how a laminar separation bubble forms and why it matters for small aircraft.
11. Sketch the camber lines of a symmetric, a positively cambered and a reflexed section, and state the sign of $c_{m,ac}$ for each.
12. Starting from $M'_r = M'_{LE} + x_r N'$, derive $c_{m,2} = c_{m,1} + (h_2 - h_1)c_l$ and state the approximation involved.
13. A section has $c_{m,ac} = +0.02$ with its AC at the quarter chord. Where is its centre of pressure at $c_l = 0.4$, and in which direction does it move as c_l rises? (Answer: 0.20 chord; aft, toward the AC.)
14. Explain the Kutta condition and its role in making the predicted lift unique.
15. Using thin-airfoil theory, show that the aerodynamic centre of any thin airfoil is at the quarter chord.
16. For the camber line with $dz/dx = B_0 + B_1 \cos \theta + B_2 \cos 2\theta$, find the reflex coefficient B_2 needed to give $c_{m,ac} = +0.03$ when $B_1 = 0.06$. (Answer: 0.098 .)
17. From a set of polars, explain how you would extract the zero-lift angle, the lift-curve slope and $c_{m,ac}$ of a section.
18. Derive $(L/D)_{\max} = 1/(2\sqrt{KC_{D0}})$ from the parabolic polar.
19. Compute the mean aerodynamic chord, its spanwise position and the estimated aerodynamic-centre position of a wing with $b = 2.0 \text{ m}$, $c_r = 0.35 \text{ m}$, $c_t = 0.14 \text{ m}$ and $\Lambda_{LE} = 30^\circ$. (Answer: $\bar{c} = 0.260 \text{ m}$ at $y_{mac} = 0.429 \text{ m}$; AC 0.312 m behind the apex.)
20. Explain, using Helmholtz's second theorem, why a change of bound circulation along the span must be accompanied by trailing vorticity.
21. Show that the elliptical lift distribution gives uniform downwash and $C_{D,i} = C_L^2/(\pi AR)$.
22. Using the monoplane-equation result for $C_{D,i}$, explain why every non-elliptical distribution has more induced drag for the same span and lift.

23. Explain the difference between basic and additional span loading and why washout gives a swept wing a nose-up moment at zero lift.
24. Explain how the induced angle tilts the local lift vector, and what happens to that tilt where a section sits in upwash.
25. Describe the vortex-lattice method and explain why the collocation point is placed at three-quarter chord.
26. Why does aft sweep promote tip stall? Give two aerodynamic reasons.
27. Derive the neutral point of a conventional aircraft and show that it reduces to the wing aerodynamic centre when the tail is removed.
28. Explain adverse yaw and why it is a particular problem for a flying wing without a fin.
29. Compute the root bending moment of one semispan for elliptical and bell-shaped loadings of total lift 20 N on a 1.6 m span. (Answer: 3.40 N·m and 2.72 N·m.)
30. A flying wing has its neutral point at 25 percent MAC and a 0.233 m MAC. By how much must a 0.45 kg battery in a 2.0 kg aircraft move to change the static margin from 8 to 10 percent? (Answer: 20.7 mm forward.)

20. How This Module Maps to the Flying-Wing Module

Flying-wing module topic	Prerequisites in this module
1. Benefits: drag, L/D, Breguet range	Sections 5, 6, 12 (drag breakdown, wetted area, Breguet equations)
1. Benefits: structural weight, span loading	Section 17 (bending moment, span loading)
1. Challenges: trim drag, stability without a tail, coupling	Sections 12.1, 16.5, 16.7–16.9
2. Resultant force, N, A, L, D	Section 8.1–8.2
2. Centre of pressure, aerodynamic centre, neutral point	Sections 8.3–8.5, 10.4, 16.9
3.1 Stability condition, static margin	Sections 1.2–1.3, 8.3, 16.7–16.9
3.2 Centre-of-pressure travel	Sections 1.2 (product and quotient rules), 8.5
3.3–3.4 Trim equation, reflexed airfoils, VLM	Sections 7.4, 10.6, 11.2, 14.12
3.5, 5.1 Washout as a tail; sweep moment arm	Sections 13.4, 14.10, 15.3
4.1 Lifting-line background, induced angle	Sections 9.5, 14.3–14.6
4.2 Elliptical distribution, minimum induced drag	Sections 1.7, 14.7–14.8
4.3 Bell-shaped distribution, Fourier derivation	Sections 1.7, 14.7, 17.2
4.4–4.5 Induced thrust, proverse yaw	Sections 14.5, 16.2, 16.5, 16.8
5.2 Geometric and aerodynamic twist, zero-lift angles	Sections 10.4, 11.1, 14.6
5.3 Sizing the twist	Sections 13, 14.10, 15.3

Flying-wing module topic	Prerequisites in this module
5.4 Taper ratio, induced drag, roll inertia, wave drag	Sections 2.5, 13, 14.9, 15.1
5.5 Tip stall, Reynolds number, stall strips	Sections 5.3, 6.3–6.6, 11.4–11.5, 14.9, 15.2

Table 4. Where each topic of the flying-wing module draws on this module.

21. Further Reading

The following texts develop the material of this module in greater depth and are standard references. J. D. Anderson, *Fundamentals of Aerodynamics*, McGraw-Hill, is the primary undergraduate text for Sections 3 to 10 and 14, and its sign conventions are those used here. J. D. Anderson, *Introduction to Flight*, McGraw-Hill, covers the atmosphere, performance and an introduction to stability at a gentler level. I. H. Abbott and A. E. von Doenhoff, *Theory of Wing Sections*, Dover, contains thin-airfoil theory and the NACA airfoil data. J. Katz and A. Plotkin, *Low-Speed Aerodynamics*, Cambridge University Press, treats panel and vortex-lattice methods in full. F. M. White, *Viscous Fluid Flow*, McGraw-Hill, is the standard reference for boundary layers. D. P. Raymer, *Aircraft Design: A Conceptual Approach*, AIAA, covers drag estimation, planform geometry and weights. B. Etkin and L. D. Reid, *Dynamics of Flight: Stability and Control*, Wiley, and R. C. Nelson, *Flight Stability and Automatic Control*, McGraw-Hill, cover axis systems, stability derivatives and the dynamic modes. K. Nickel and M. Wohlfahrt, *Tailless Aircraft in Theory and Practice*, AIAA and Butterworth-Heinemann, is the classic book on flying wings and links this module directly to the companion module. A. H. Bowers and co-authors, *On Wings of the Minimum Induced Drag: Spanload Implications for Aircraft and Birds*, NASA Technical Publication, 2016, describes the bell-shaped spanload and the flight experiments on proverse yaw. M. Drela's XFOIL and AVL and A. Deperrois's XFLR5 are freely available programs implementing the panel, boundary-layer and vortex-lattice methods described here.